Stability of restrictions of cotangent bundles of irreducible Hermitian symmetric spaces of compact type

Indranil Biswas, Pierre-Emmanuel Chaput, Christophe Mourougane

STABILITY OF RESTRICTIONS OF COTANGENT BUNDLES OF IRREDUCIBLE HERMITIAN SYMMETRIC SPACES OF COMPACT TYPE

INDRANIL BISWAS, PIERRE-EMMANUEL CHAPUT, AND CHRISTOPHE MOUROUGANE

ABSTRACT. It is known that the cotangent bundle Ω_Y of an irreducible Hermitian symmetric space Y of compact type is stable. Except for a few obvious exceptions, we show that if $X \subset Y$ is a complete intersection such that $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is surjective, then the restriction $\Omega_{Y|X}$ is stable. We then address some cases where the Picard group increases by restriction.

CONTENTS

1. INTRODUCTION

Stable vector bundles with zero characteristic classes on a smooth projective variety are given by the irreducible unitary representations of the fundamental group of the variety. On the other hand, it is a difficult

Date: April 1, 2015.
2000 *Mathematics Subject Classification.* 14J60, 14M17.

and interesting question to produce explicit examples of stable vector bundles on algebraic varieties with non-zero characteristic classes. However, there are such vector bundles within the framework of homogeneous spaces. For example, an irreducible homogeneous bundle E on a homogeneous space Y is stable [Ume78], [Ram66], [Bis04]. Once one has these stable vector bundles, a theorem of Mehta-Ramanathan [MR84] and Flenner [Fle84] asserts the stability of restrictions of these bundles to general hypersurfaces X of high enough degree with respect to the given polarisation. In this article, we address the following general question: What can be said about the stability of the restriction of an irreducible homogeneous bundle E, defined on a homogeneous space Y, to a subvariety $X \subset Y$?

We give a positive answer to this question in the following setting. Recall that a Hermitian symmetric space is a Hermitian manifold in which every point is an isolated point of an isometric involution. It is homogeneous under its isometry group. Compact examples include the usual Grassmannians, the quadric hypersurfaces, the Lagrangian Grassmannians parametrising n-dimensional Lagrangian subspaces of $\mathbb{C}^{2n}$ equipped with a symplectic form, the spinor Grassmannian parametrising one family of n-dimensional isotropic subspaces of $\mathbb{C}^{2n}$ equipped with a nondegenerate quadratic form, and two exceptional manifolds. For a compact irreducible Hermitian symmetric spaces Y, the cotangent bundle Ω_Y is an irreducible homogeneous, hence stable, vector bundle.

We assume first that Y is a compact Hermitian symmetric space, and X is a locally factorial complete intersection such that the restriction $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is surjective. This holds by Lefschetz theorem [Lef21] whenever $\dim X \geq 3$ (see also [Laz04, Example 3.1.25]) or whenever X is a very general complete intersection surface in $\mathbb{P}^n$ except if it is a degree $d \leq 3$ surface in $\mathbb{P}^3$ or the intersection of two quadric threefolds in $\mathbb{P}^4$ (see also [Kim91, Theorem 1]). Furthermore, we assume that E is the cotangent (or, dually, the tangent) bundle of Y.

Theorem A (Theorem 1 and Theorem 2)**.** *Let Y be a compact irreducible Hermitian symmetric space, and let X be a locally factorial positive dimensional complete intersection in Y. Assume that the restriction homomorphism $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is surjective. If Y is a projective space or a quadric, assume moreover that X has no linear equation. Then, the restriction of Ω_Y to X is stable.*

In fact, our theorems are slightly more general than Theorem A since it is not necessary that X is a complete intersection : it is enough that X has a *short resolution*, as explained in Definition 2.1. Moreover, if one is interested in general complete intersections, then using a *relative* Harder-Narasimhan filtration one can get rid of the assumption on the Picard groups ; but semistability, instead of stability, is obtained (see Theorem 4).

We then deal with the case of small dimensions where the Picard group increases. Recall that irreducible Hermitian symmetric spaces of dimension 2 or 3 are $\mathbb{P}^2$, $\mathbb{P}^3$, $\mathbb{Q}^2$ and $\mathbb{Q}^3$.

Theorem B (Theorem 4, Theorem 5)**.** *Let Y be a compact irreducible Hermitian symmetric space of dimension 2 or 3. Let $X \subset Y$ be a smooth divisor, and in the case of ambient dimension 3 let C be a complete intersection curve.*

- *Take $Y = \mathbb{P}^2$. Then $\Omega_{Y|X}$ is semi-stable if $\deg X \geq 2$. If $\deg X \geq 3$, then $\Omega_{Y|X}$ is stable.*
- *Take $Y = \mathbb{P}^3$. Then $\Omega_{Y|X}$ is stable if $\deg X \geq 2$. If C is cut-out by general non-linear hypersurfaces. Then $\Omega_{Y|C}$ is semi-stable.*
- *If $Y = \mathbb{Q}^2$, then $\Omega_{Y|X}$ is semi-stable but not stable.*

- *Take $Y = \mathbb{Q}^3$.*
  Assume that $\deg X = 1$. Then $\Omega_{Y|X}$ is semi-stable.
  Assume that $\deg X = 2$. Then $\Omega_{Y|X}$ is stable.
  Assume that $\deg X \geq 9$. Then $\Omega_{Y|X}$ is stable.
  If $\deg X \geq 3$ and X is very general, then $\Omega_{Y|X}$ is stable.
  If C is cut-out by general non-linear hypersurfaces, then $\Omega_{Y|C}$ is semi-stable.

Our arguments are very different in the two situations. In the case of complete intersections with no increase of Picard groups, we use a new vanishing theorem (Theorem 3) that may be of independent interest (see Section 3.1). In fact, by standard cohomological arguments, a subbundle $\mathcal{F} \subset \Omega_{Y|X}$ of the restriction of Ω_Y contradicting stability yields the non-vanishing of some cohomology group $H^q(Y, \Omega^p(l))$, where q is related to the codimension of X and l to the degree of $\mathcal{F}$. Our vanishing theorem implies the desired stability inequality (see Section 2). In small dimensions, we use tools from projective geometry to make explicit the new line bundles that appear on the subvariety X and some arguments of representation theory (see Section 4).

2. RESTRICTIONS WITH SMALL PICARD GROUP

Let Y be a compact irreducible Hermitian symmetric space, but not isomorphic to $\mathbb{Q}^2$. We denote the ample generator of the Picard group of Y by $\mathcal{O}_Y(1)$. For a sheaf $\mathcal{F}$ on Y and an integer l, we denote the tensor product $\mathcal{F} \otimes \mathcal{O}_Y(1)^{\otimes l}$ by $\mathcal{F}(l)$. We denote by $\deg Y$ the top degree self-intersection $\deg Y := \mathcal{O}_Y(1)^{\dim Y}$ and by $c_1(Y)$ the index of the Fano manifold Y. We recall that $c_1(Y)$ is defined by the equality $-K_Y = c_1(Y)\mathcal{O}_Y(1)$ as elements of the Néron-Severi group of Y.

2.1. **Short resolution.** We will prove stability of the restriction of Ω_Y to subschemes whose structure sheaf has a short resolution in the following sense :

Definition 2.1. A subscheme $X \subset Y$ is said to have a *short resolution* if there is a resolution

$$0 \to \mathcal{F}_k \to \mathcal{F}_{k-1} \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{O}_X \to 0\,,$$

where $\mathcal{F}_0 = \mathcal{O}_Y$, $\mathcal{F}_i := \oplus_j \mathcal{O}_Y(-d_{ij})$ with $d_{ij} \geq i$ for $i > 0$, and the length k of the resolution satisfies $k < \dim(Y)$.

Example 2.2. *The Koszul resolution of complete intersections is a short resolution for a positive-dimensional complete intersection in Y. If, moreover, none of the equations is linear, then the integers d_{ij} in Definition 2.1 satisfy $d_{ij} \geq i + 1$.*

Our second class of examples are some arithmetically Cohen-Macaulay subschemes. Let $X \subset \mathbb{P}^N$ be a subscheme defined by a homogeneous ideal J in the homogeneous coordinate ring $A := \mathbb{C}[X_0, \ldots, X_N]$. Recall that X is called *arithmetically Cohen-Macaulay* if the depth of A/J is equal to the dimension of A/J, namely $\dim X + 1$. Let, moreover, $I \subset A$ denote the homogeneous ideal of$Y$. The reason why we will consider arithmetically Cohen-Macaulay subschemes is the following :

Fact 2.3. *Let $X \subset Y \subset \mathbb{P}^N$ be an arithmetically Cohen-Macaulay subscheme, and assume that A/J has finite projective dimension over A/I. Then, the structure sheaf $\mathcal{O}_X$ has a resolution by split vector bundles over Y of length $k = \dim Y - \dim X$*

$$0 \to \mathcal{F}_k \to \mathcal{F}_{k-1} \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{O}_X \to 0\,,$$

where $\mathcal{F}_0 = \mathcal{O}_Y$ and $\mathcal{F}_i := \oplus_j \mathcal{O}_Y(-d_{ij})$ with $d_{ij} \geq i$ for $i > 0$. In particular, if $\dim(X) > 0$, then X has a short resolution.

If, moreover, X is not linearly degenerate in $\mathbb{P}^N$, then we have $d_{ij} \geq i + 1$.

Proof. We have $I \subset J \subset A$. By Auslander-Buchsbaum formula [Mat89, Theorem 19.1], we have the equality

$$\mathrm{pd}_{A/I}(A/J) = \mathrm{depth}(A/I) - \mathrm{depth}(A/J)\,.$$

Moreover, any homogeneous space embedded by a homogeneous ample line bundle is arithmetically Cohen-Macaulay (see for example [BK05, Corollary 3.4.4]). Thus,

$$\mathrm{depth}(A/I) = \dim(A/I) = \dim Y + 1\,.$$

Therefore, $\mathrm{pd}_{A/I}(A/J) = \dim Y - \dim X$. Hence a minimal free resolution of A/J over A/I has length $k = \dim Y - \dim X$. Moreover, since for such a resolution the differentials have positive degree, we have $d_{ij} \geq i$ for all $\forall i, j$. If X is not included in any hyperplane, it has no equation of degree 1, so $d_{1j} \geq 2$ for all $\forall j$, and we deduce that $d_{ij} \geq i + 1$. □

2.2. **A cohomological property.** We will need the following nonvanishing property to prove our stability results. Its proof is postponed to Section 3.

Proposition 2.4. *Let Y be a compact irreducible Hermitian symmetric space, but not a projective space. Let l, p, q be integers, with $q < \dim Y$, such that $H^q(Y, \Omega_Y^p(l)) \neq 0$. Then,*

$$l + q \;\geq\; p\frac{c_1(Y)}{\dim(Y)}\,,$$

with equality holding if and only if

- *$p = \dim(Y), q = 0$ and $l = c_1(Y)$,*
- *or $p = q = 0$ and $l \geq 0$,*
- *or Y is a quadric and $l = 0$,*
- *or $Y \simeq \mathbb{Q}^4, l = 2, p = 3, q = 1$.*

Of course, if $q = \dim(Y)$ were allowed, then the above inequality would fail, since we may have l very negative and $p = \dim(Y)$ (then apply Serre duality).

2.3. **General argument.** We can now prove the

Theorem 1. *Let Y be any compact irreducible Hermitian symmetric space excluding a projective space and a quadric. Let X be a locally factorial positive dimensional subvariety of Y having a short resolution and such that $\mathrm{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_Y(1)_{|X}$. Then the restriction of Ω_Y to X is stable.*

Note that if $\dim X \geq 3$, the constraint on the Picard group of the complete intersection is ensured by Lefschetz theorem.

Proof. We will later prove a slightly weaker result for quadrics and projective spaces (Theorem 2), thus, for the moment Y is any compact irreducible Hermitian symmetric space.

Let $\mathcal{F}$ be a coherent subsheaf of $\Omega_{Y|X}$ of rank $0 < p < \dim Y$. Since X is assumed to be locally factorial, the rank one reflexive subsheaf $\det \mathcal{F} := (\bigwedge^p \mathcal{F})^{\star\star}$ of $\bigwedge^p \Omega_{Y|X}$ is invertible [Har80, Proposition

1.9] and hence isomorphic to $\mathcal{O}_X(-d)$ for some integer d. We have

$$\begin{aligned}\mu(\Omega_Y) &= \frac{\mathcal{O}_X(1)^{\dim(X)-1}\cdot K_Y}{\operatorname{rank}\Omega_{Y|X}} = -\frac{c_1(Y)}{\dim(Y)}\cdot \deg X\cdot \deg Y\\ \mu(\mathcal{F}) &= \frac{\mathcal{O}_X(1)^{\dim(X)-1}\cdot \det\mathcal{F}}{\operatorname{rank}\mathcal{F}} = -\frac{d}{p}\cdot \deg X\cdot \deg Y\,.\end{aligned}$$

The inclusion $\mathcal{F}\subset \Omega_{Y|X}$ yields the non-vanishing of

$$H^0(X, Hom(\det\mathcal{F}, \Omega^p_{Y|X})) \;=\; H^0(X, \Omega^p_Y(d)_{|X}),$$

from which we have to deduce the stability inequality

$$\mu(\mathcal{F}) \;<\; \mu(\Omega_{Y|X})\,,\;\; \textit{equivalently},\;\; d \;>\; p\,\frac{c_1(Y)}{\dim(Y)}.$$

Consider a resolution of $\mathcal{O}_X$ as in Definition 2.1. The resolution

$$0\to \mathcal{F}_k\otimes\Omega^p_Y(d)\to\cdots\to\mathcal{F}_1\otimes\Omega^p_Y(d)\to\mathcal{F}_0\otimes\Omega^p_Y(d)\to\Omega^p_Y(d)_{|X}\to 0$$

translates the non-vanishing of $H^0(X,\Omega^p_Y(d)_{|X})$ into the nonvanishing of one of the cohomology groups in the decomposition

$$H^i(Y,\mathcal{F}_i\otimes\Omega^p_Y(d)) = \oplus_j H^i(Y,\Omega^p_Y(d-d_{ij})),$$

say of $H^i(Y,\Omega^p_Y(d-d_{ij}))$.

We now assume that Y is not a projective space. In our setting Proposition 2.4 reads

$$(d-d_{ij}) + i \geq p\frac{c_1(Y)}{\dim(Y)}. \tag{2.1}$$

It follows that $d\geq p\frac{c_1(Y)}{\dim(Y)}$.

In case of the equality $d = p\frac{c_1(Y)}{\dim(Y)}$, we get that $d_{ij}=i$, and the equality in (2.1) holds. Now assume that Y is not a quadric. Therefore, Proposition 2.4 gives that $p=0$ or $p=\dim(Y)$, equivalently, either $\mathcal{F}=\{0\}$ or $\mathcal{F}=\Omega_Y$. Thus, Ω_Y is stable. □

Some remarks regarding the two excluded cases in Theorem 1.

Remark 2.5. If $X\subset Y$ is a linear section of the quadric Y, then, as the above proof shows, the restriction of Ω_Y to X is semi-stable.

Remark 2.6. If $X\subset Y$ is a smooth quadric inside the quadric Y, then we have an exact sequence

$$0\to N^*_{X|Y}\to\Omega_{Y|X}\to\Omega_X\to 0.$$

All these vector bundles have equal slope, so $\Omega_{Y|X}$ is not stable. We believe that in this situation the only destabilizing subsheaf is $N^*_{X|Y}$. The above argument shows that a destabilizing sheaf must have rank equal to the codimension of X.

Remark 2.7. Similarly, if $X\subset Y$ is contained in a linear subspace H in a projective space Y, then for the exact sequence,

$$0\to N^*_{H|Y_{|X}}\to\Omega_{Y|X}\to\Omega_{H|X}\to 0\,,$$

the slope of $N^*_{H|Y_{|X}}$ is strictly bigger than the slope of $\Omega_{Y|X}$. Thus, $\Omega_{Y|X}$ is not even semi-stable.

Thus, to get a result similar to Theorem 1 in these two cases, we exclude the case where X has a linear equation :

Theorem 2. *Let Y be a smooth quadric or a projective space. Then Ω_Y is stable. Let X be a locally factorial subvariety in Y having a short resolution and such that $\mathrm{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_Y(1)_{|X}$. Assume that X is contained in no hyperplane section of Y. Then the restriction of Ω_Y to X is stable.*

Note that by Lefschetz theorem, this theorem applies to very general cubic hypersurfaces of $\mathbb{Q}^3$.

Proof. We continue with the notation of Theorem 1. If Y is a quadric, by the above proof of Theorem 1, we have the non-vanishing of some $H^i(Y, \mathcal{F}_i \otimes \Omega_Y^p(d))$. If $i > 0$, by (2.1) we have for some j the inequality

$$d - d_{ij} + i \geq p \frac{c_1(Y)}{\dim(Y)} \, (= p) \, .$$

Since, by Fact 2.3, $d_{ij} > i$, we get that $d > p$. If $i = 0$, then $H^0(Y, \Omega_Y^p(d)) \neq 0$, and by a result due to Snow (see Section 3.3), we get that either $p = 0$ or $p = \dim(Y)$. This implies stability as in the proof of Theorem 1.

Assume now that Y is the projective space $\mathbb{P}^n$. We may assume that $0 < p < n$. We wish to prove that

$$\frac{d}{p} > \frac{n+1}{n} \, .$$

Since $\frac{p+1}{p} > \frac{n+1}{n}$, it is enough to prove that $d \geq p + 1$. For integers p, q, l, we have $H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(l)) \neq 0$ if and only if one of the following hold :

(1) $l > 0, p < l$ and $q = 0$,
(2) $l = 0$ and $p = q$,
(3) $l < 0, n - p < -l$ and $q = n$.

Once again, we get that $H^i(Y, \mathcal{F}_i \otimes \Omega_Y^p(d)) \neq 0$ for some i. If $i = 0$, since $\mathcal{F}_0 = \mathcal{O}_Y$, this implies that $p = 0$. If $i > 0$, since $i < n$, this implies that $i = p$ and $d = d_{ij}$ for some j. Thus we have $d \geq i + 1 = p + 1$, as claimed. □

3. Vanishing theorems

3.1. **The statement.** We will now explain the proof of Proposition 2.4. Actually, we will prove a stronger vanishing theorem that may be useful for other purposes.

Theorem 3. *Let Y be a compact irreducible Hermitian symmetric space, but not a projective space. Let l, p, q be integers, with $l > 0$, $p > 0$, such that $H^q(Y, \Omega_Y^p(l)) \neq 0$.*

(1) *Then,*

$$l + q \geq p \frac{c_1(Y)}{\dim(Y)}.$$

Furthermore, equality holds if and only if
- $p = \dim(Y), q = 0$ *and* $l = c_1(Y)$,
- *or Y is a quadric and $l = 0$,*
- *or $Y \simeq \mathbb{Q}^4, l = 2, p = 3, q = 1$.*

(2) *If moreover $q > 0$, then*

$$l + q \leq p.$$

The proof of Theorem 3 is given in the next subsections. Surprisingly enough, the first item is very intricate, and the proof of the vanishing theorem in this case entails involved combinatorial arguments. The second item is much easier.

Let us explain that Theorem 3 implies Proposition 2.4.

Assume that $H^q(Y,\Omega_Y^p(l)) \neq 0$. If $l=0$, then by Hodge theory we have $p=q$. Since $c_1(Y) \leq \dim(Y)$ with equality holding if and only if Y is a quadric [KO73, page 37], this case is settled. If $p = 0$, then $q = 0$ if $l > 0$, while $q = \dim(Y)$ if $l < 0$. Thus this case is also settled.

In the second case of the theorem, let us prove that actually

$$l+q \,<\, p\frac{c_1(Y)}{\dim(Y)} + \dim(Y) - c_1(Y)\,. \tag{3.1}$$

Firstly, the theorem states that $l+q \,\leq\, p$, and we have $p \leq \dim(Y)$. Since

$$(1-\frac{c_1(Y)}{\dim(Y)})(l+q) \leq \dim(Y) - c_1(Y)\,,$$

we have

$$l+q \leq \frac{c_1(Y)}{\dim(Y)}(l+q) + \dim(Y) - c_1(Y) \leq p\frac{c_1(Y)}{\dim(Y)} + \dim(Y) - c_1(Y)\,.$$

Note that the equality here would imply that $p = \dim(Y)$, in which case we cannot have $q > 0$. Thus, the inequality (3.1) is proved. Now, coming back to the proof of the theorem, if $l < 0$, then by Serre duality we have

$$H^{\dim(Y)-q}(Y,\Omega_Y^{\dim(Y)-p}(-l)) \,\neq\, 0\,.$$

The relation (3.1) gives that $(-l) + (\dim(Y)-q) < (\dim(Y)-p)\frac{c_1(Y)}{\dim(Y)} + \dim(Y) - c_1(Y)$, or in other words,

$$l+q \,>\, p\frac{c_1(Y)}{\dim(Y)}\,.$$

3.2. **The case of Grassmannians (type A_n).** Fix positive integers $a,b \geq 2$. Let $\mathrm{G}_{ab} := G(a,a+b)$ be the Grassmannian that parametrises a-dimensional linear subspaces of a fixed $(a+b)$-dimensional k-vector space V. It is the homogeneous space $G(a,a+b) = \mathrm{SU}(a+b)/[\mathrm{SU}(a+b)\cap \mathrm{U}(a)\times \mathrm{U}(b)]$. Let $\mathcal{O}_{ab}(1)$ be the Plücker polarisation on G_{ab}, which is also the positive generator of $\mathrm{Pic}(\mathrm{G}_{ab})$. The cotangent bundle Ω_{ab} of G_{ab} being homogeneous and irreducible is μ-stable. We assume $\dim \mathrm{G}_{ab} = ab \geq 4$ so that by Lefschetz hyperplane theorem, the restriction $\mathcal{O}_{ab}(1)_{|X}$ generates the Picard group of every smooth hypersurface X of G_{ab}. Theorem 1 reads in this case as follows :

Proposition 3.1 (Theorem 1 for Grassmannians)**.** *Assume that $ab \geq 4$. Let X be a smooth complete intersection in G_{ab} such that $\mathrm{Pic}(X) = \mathbb{Z}\cdot\mathcal{O}_{ab}(1)_{|X}$. Then the restriction of Ω_{ab} to X is semi-stable. It is stable except exactly when $a=b=2$ with X being a hyperplane section.*

In case Y is a Grassmannian, by [Sno86], the non-vanishing of $H^q(Y,\Omega_Y^p(l))$ implies the existence of a partition of p, which is l-admissible with cohomological degree q in the following sense.

Definition 3.2 ([Sno86])**.** Let λ be a partition and l an integer. We say that λ is *l-admissible* if no hook-number of λ is equal to l. The (l-)*cohomological degree* of an l-admissible partition is the number of hook-numbers which are greater than l.

Hence, as we have seen in the proof of Theorem 1, the above Proposition 3.1 is a consequence of the following combinatorial result :

Proposition 3.3 (First part of Theorem 3 for Grassmannians)**.** *Let l be a non-negative integer. Let λ be a l-admissible partition of p in a rectangle $a \times b$, with cohomological degree q. Then, we have the inequality*

$$l + q \geq p\frac{a+b}{ab},$$

with equality holding if and only if either $\lambda = (b^a)$, $l = a + b$ and $q = 0$, or $\lambda = (2, 1)$, $l = 2$ and $q = 1$.

Proof. The case where $l = 0$ is easy as then we have $p = q$, and if $a\,, b \geq 2$, the inequality $ab \geq a + b$ holds.

$\sum_{i=1}^{n} x_i + n-1$		$\sum_{i=2}^{n} x_i + n$	$\sum_{i=2}^{n} x_i + n-2$		$\sum_{i=3}^{n} x_i + n-1$	$\sum_{i=3}^{n} x_i + n-3$	$\dots$	$\sum_{n-k+1}^{n} x_i + k+1$	$\sum_{n-k+1}^{n} x_i + k-1$	$\sum_{n-3}^{n} x_i + 3$		$\sum_{n-2}^{n} x_i + 4$	$\sum_{n-2}^{n} x_i + 2$		$x_{n-1} + x_n + 3$	$x_{n-1} + x_n + 1$		$x_n + 2$	x_n	1
$\sum_{i=1}^{n-1} x_i + n-2$		$\sum_{i=2}^{n-1} x_i + n-1$	$\sum_{i=2}^{n-1} x_i + n-3$		$\sum_{i=3}^{n-1} x_i + n-2$	$\sum_{i=3}^{n-1} x_i + n-4$	$\dots$	$\dots$	$\dots$	$\sum_{n-3}^{n-1} x_i + 2$		$x_{n-1} + x_{n-2} + 3$	$x_{n-1} + x_{n-2} + 1$		$x_{n-1} + 2$	x_{n-1}		1		
$\sum_{i=1}^{n-2} x_i + n-3$		$\sum_{i=2}^{n-2} x_i + n-2$	$\sum_{i=2}^{n-2} x_i + n-4$		$\sum_{i=3}^{n-2} x_i + n-3$	$\sum_{i=3}^{n-2} x_i + n-5$	$\dots$	$\dots$	$\dots$	$x_{n-3} + x_{n-2} + 1$		$x_{n-2} + 2$	x_{n-2}		1					
$\sum_{i=1}^{n-3} x_i + n-4$		$\sum_{i=2}^{n-3} x_i + n-3$	$\sum_{i=2}^{n-3} x_i + n-5$		$\sum_{i=3}^{n-3} x_i + n-4$	$\sum_{i=3}^{n-3} x_i + n-6$	$\dots$	$\dots$	$\dots$	x_{n-3}		1								
$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$														
$\sum_{j=1}^{k+2} x_j + k+1$		$\sum_{j=2}^{k+2} x_j + k+2$	$\sum_{j=2}^{k+2} x_j + k$		$\sum_{j=3}^{k+2} x_j + k+1$	$\sum_{j=3}^{k+2} x_j + k-1$														
$\sum_{j=1}^{k+1} x_j + k$		$\sum_{j=2}^{k+1} x_j + k+1$	$\sum_{j=2}^{k+1} x_j + k-1$																	
$\sum_{j=1}^{k} x_j + k-1$		$\sum_{j=2}^{k+1} x_j + k+1$	$\sum_{j=2}^{k+1} x_j + k-1$					x_k	1											
$\vdots$		$\vdots$	$\vdots$																	
$x_1 + x_2 + x_3 + 2$		$x_2 + x_3 + 3$	$x_2 + x_3 + 1$		$x_3 + 2$	x_3	1													
$x_1 + x_2 + 1$		$x_2 + 2$	x_2		1															
x_1		1																		

We choose to denote partitions by non-decreasing sequences $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{b'})$. Let $x_1 = \lambda_1$ be the first part; for all $2 \leq i \leq n$, let x_i denote the increase $\lambda_i - \lambda_{i-1}$ between consecutive parts. We denote by q_i the number of boxes in the i^{th} row strictly bigger than l. From the reading of the top row, we obtain that the gap l has the expression

$$l = \sum_{j=i}^{n} x_j + (n+1-i)$$

for some index i with $2 \leq i \leq n$. If $x_{n-2} = 0$ for example, then the whole block

x_{n-2} $+x_{n-1}$ $+x_n+2$		x_{n-1} $+x_n$ $+3$
x_{n-1} $+x_{n-2}$ $+1$		x_{n-1} $+2$
x_{n-2}		1

gets removed, and we get two consecutive values $x_{n-1} + x_n + 2$ and

$$x_{n-2} + x_{n-1} + x_n + 3 \;=\; x_{n-1} + x_n + 3$$

as candidates for gaps on the top row. For example, with $x_1 = 2, x_2 = 0, x_3 = 2, x_4 = 2$, we get 6 and 7 as candidates for gaps on the top row.

9	8	5	4	2	1
6	5	2	1		
3	2				
2	1				

The number of boxes strictly bigger than l on the top row is $q_n = \sum_{j=1}^{i-1} x_j$. We have

$$l + q = \sum_{j=1}^{n} x_j + (n+1-i) + \sum_{j=1}^{n-1} q_j = b' + a' + 1 - i + \sum_{j=1}^{n-1} q_j.$$

On the other hand, counting boxes in the complementary partition, we have

$$a' - \frac{p}{b'} \quad = \quad = \frac{a'b' - p}{b'} = \frac{\sum_{j=2}^{n} (j-1) x_j}{\sum_{j=1}^{n} x_j}.$$

Up to changing the fixed vector space V by its dual, we may assume that $a' \leq b'$, so that $b' - \frac{p}{a'} \geq a' - \frac{p}{b'}$. We have to show that $l + q \geq \dfrac{p}{a'} + \dfrac{p}{b'}$, that is $(a' - \dfrac{p}{b'}) + (b' - \dfrac{p}{a'}) \geq i - 1 - \displaystyle\sum_{j=1}^{n-1} q_j$. It is enough to prove that

$$a' - \frac{p}{b'} \;\geq\; \frac{i - 1 - \sum_{j=1}^{n-1} q_j}{2}. \qquad (3.2)$$

Let $k = \#\{i | c_{i,1} < l\}$, where $c_{i,1}$ denotes the hook number in the $(i,1)$-box. We have

$$(\sum_{j=1}^{k} x_j) + k - 1 < l \leq (\sum_{j=1}^{k+1} x_j) + k$$

(see the red boxes). As $l \leq (\sum_{j=1}^{k+1} x_j) + k$, the number l is one of the gaps in the $(k+1)^{th}$ row, and hence, $l \leq (\sum_{j=2}^{k+1} x_j) + k$ (see the green boxes). As $l \leq (\sum_{j=2}^{k+2} x_j) + k$ (see the blue box), it is a gap in the $(k+2)^{th}$ row with $l \leq (\sum_{j=3}^{k+2} x_j) + k$ (see the green boxes). Arguing further, we get that on the n^{th} row, $l \leq (\sum_{j=n-k+1}^{n} x_j) + k$ forcing the index i to satisfy the inequality

$$i \geq n - k + 1.$$

We also infer that on each row above the k^{th}, there are boxes strictly bigger than l, so that considering the number of boxes strictly bigger than l on non-top rows we have $\sum_{j=1}^{n-1} q_j \geq n - k - 1$, in particular

$$i - 1 - \sum_{j=1}^{n-1} q_j \leq k\,.$$

From the choice of the level k we have $(\sum_{j=1}^{k} x_j) + k \leq l = \sum_{j=i}^{n} x_j + (n + 1 - i)$. Now cancelling the common terms in these sums, we infer the following bound on the first increases in terms of the last ones

$$\sum_{j=1}^{\min(k,i-1)} x_j \leq \sum_{j=\max(k+1,i)}^{n} x_j + (n - k + 1 - i) \leq \sum_{j=\max(k+1,i)}^{n} x_j. \tag{3.3}$$

Therefore

$$\begin{aligned} a' - \frac{p}{b'} &= \frac{\sum_{j=2}^{n}(j-1)x_j}{\sum_{j=1}^{n} x_j} = \frac{\sum_{j=2}^{n}(j-1)x_j}{\sum_{j=1}^{\min(k,i-1)} x_j + \sum_{\min(k+1,i)}^{n} x_j} \\ &\geq \frac{\sum_{j=2}^{n}(j-1)x_j}{\sum_{j=\max(k+1,i)}^{n} x_j + \sum_{\min(k+1,i)}^{n} x_j} \qquad (3.4) \\ &= \frac{\sum_{j=2}^{n}(j-1)x_j}{\sum_{j=\min(k+1,i)}^{\max(k+1,i)-1} x_j + 2\sum_{\max(k+1,i)}^{n} x_j} \\ &\geq \frac{\min(k, i-1)\sum_{j=\min(k+1,i)}^{n} x_j}{2\sum_{j=\min(k+1,i)}^{n} x_j} \qquad (3.5) \\ &= \frac{\min(k, i-1)}{2} \geq \frac{i - 1 - \sum_{j=1}^{n-1} q_j}{2}. \end{aligned}$$

This proves the inequality for semi-stability.

If the inequality in (3.5) is not strict, then comparing numerators, we have

$$x_2 = x_3 = \cdots = x_{\min(k,i-1)} = 0 \text{ and } x_{\min(k+2,i+1)} = \cdots = x_n = 0\,.$$

The partition is not a rectangle, so that $x_{\min(k+1,i)} \neq 0$.

$x_1+x_{k+1}+n-1$	$x_1+x_{k+1}+n-2$	$\cdots$	$x_{k+1}+n$	$x_{k+1}+n-k-1$	$x_{k+1}+n-k-2$	$\cdots$	$n-k$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$x_1+x_{k+1}+k+1$	$x_1+x_{k+1}+k$	$\cdots$	$x_{k+1}+k+2$	$x_{k+1}+1$	x_{k+1}	$\cdots$	2
$x_1+x_{k+1}+k$	$x_1+x_{k+1}+k-1$	$\cdots$	$x_{k+1}+k+1$	x_{k+1}	$x_{k+1}-1$	$\cdots$	1
x_1+k-1	x_1+k-2	$\cdots$	k				
$\vdots$	$\vdots$		$\vdots$				
x_1+1	x_1	$\cdots$	2				
x_1	x_1-1	$\cdots$	1				

Comparing denominators, we find that $\min(k+1, i) = \max(k+1, i)$. So the level k satisfies the equation $k = i-1$ and $a' - \frac{p}{b'} \geq \frac{i-1}{2}$. If the inequality in (3.2) is not strict, then $\sum_{j=1}^{n-1} q_j = 0$, so the level k must also be maximal $k = n-1$.

x_1+x_n+n-1		$\cdots$		x_n+n	x_n	x_n-1		$\cdots$		2	1
x_1+n-2		$\cdots$		$n-1$							
$\vdots$				$\vdots$							
x_1+1		$\cdots$		2							
x_1		$\cdots$		1							

If the first inequality in (3.4) is not strict, then recalling (3.3) we find a third constraint on the level k, namely $n-k+1-i = 0$. This gives us $k = i-1 = n-1 = 1$. This proves the requirement (3.2) with strict inequality except in the following case:

3	1
1	

$\square$

We now prove the second part of Theorem 3.

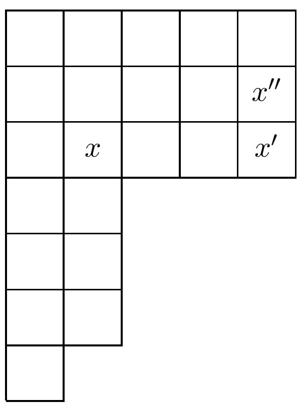
Proposition 3.4 (Second part of Theorem 3 for Grassmannians)**.** *Let l be a positive integer. Let λ be a l-admissible partition of p with l-cohomological degree $q > 0$. Then, we have the inequality*

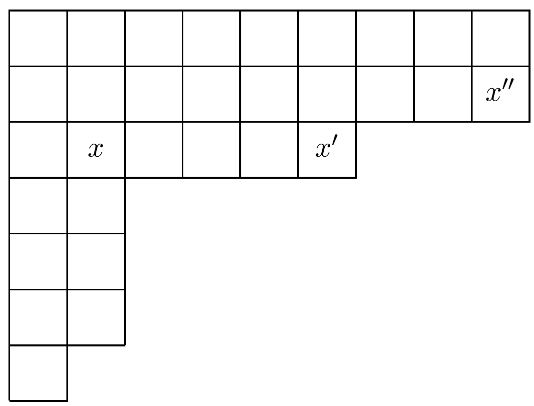
$$l + q \leq p.$$

Moreover, if the equality $p = l + q$ holds, then λ is a hook (i.e., it's shape is $(a, 1^b)$).

Proof. Let $Y(\lambda) := \{(i,j) \mid j \leq \lambda_i\} \subset \mathbb{N}^2$ be the Young diagram of λ. For $x \in Y(\lambda)$, we denote by $h(x)$ the hook number at x. We have $p = \#Y(\lambda)$ and $q = \#\{x \in Y(\lambda) \mid h(x) > l\}$. Since $q > 0$, let $x \in Y(\lambda)$ such that $h(x) > l$. Moreover, we can assume that x is minimal for this property, namely that $h(y) < l$ if y is south-east from x. By definition of $h(x)$, there are $h(x) - 1$ elements $z \in Y(\lambda)$ which are either on the same row as x on its right, or under x in the same column. For these elements, we have $h(z) < l$. This implies that $p - q = \#\{y \in Y(\lambda) \mid h(y) < l\} \geq h(x) - 1 \geq l$.

We now deal with the case of equality (that will not be used in the sequel). If the equality $p = l + q$ occurs, with $q > 0$ as above, then we first show that x is on the first row.





Case $\lambda_{i-1} = \lambda_i$ Case $\lambda_{i-1} > \lambda_i$

If x is not on the first row, then the hook number of the box x'', very right on the row over that of x not in the same row of x neither on the same column, is 2 or 1. Hence this box contribute to $\{y \in Y(\lambda) \mid h(y) < l\}$ and therefore, $p - q > l$. In the same way, we can show that x is on the first column and that all the boxes with hook number smaller than l are on the hook of x. Therefore λ is a hook. $\square$

3.3. **The case of quadrics (type B_n or D_n).** Let Y be a non singular quadric hypersurface of dimension n with its natural polarisation $\mathcal{O}_Y(1) = \mathcal{O}_{\mathbb{P}^{n+1}}(1)_{|Y}$. It is the homogeneous space

$$Y \;=\; \mathrm{SO}(n+2)/(\mathrm{SO}(n) \times \mathrm{SO}(2))\,.$$

From the adjunction formula, $c_1(Y) = n + 2 - 2 = \dim Y$. Recall a theorem of Snow.

Theorem ([Sno86, page 174])**.** *Let Y be a nonsingular quadric hypersurface of dimension n. If $H^q(Y, \Omega_Y^p(l)) \neq 0$, then*

- *either $p = q$ and $l = 0$,*
- *or $q = n - p$ and $l = -n + 2p$,*
- *or $q = 0$ and $l > p$,*
- *or $q = n$ and $l < -n + p$.*

As when $q = 0$ and $0 < p < n$, the inequality $l > p$ holds, the cotangent bundle Ω_Y of the quadric is stable as soon as the Picard group of the quadric is the restriction of that of $\mathbb{P}^{n+1}$, for example when $n \geq 3$. Theorem 3 follows for quadrics by checking the above cases.

3.4. **The case of Lagrangian Grassmannians (type C_n).** In this case, Y is the so-called Lagrangian Grassmannian, parametrising n-dimensional Lagrangian subspaces of $\mathbb{C}^{2n}$ equipped with the standard symplectic form. It is the homogeneous space $Y = \mathrm{Sp}(2n, \mathbb{C})/\mathrm{U}(n)$. By [Sno88], the non-vanishing of $H^q(Y, \Omega_Y^p(l))$ amounts to the existence of an l-admissible C_n-sequence of weight p and cohomological degree q, in the following sense :

Definition 3.5. Fix $l, n \in \mathbb{N}$ with $l > 0$. A n-uple of integers $(x_i)_{1 \leq i \leq n}$ will be called an *l-admissible C_n-sequence* if

- $\forall\, 1 \leq i \leq n, |x_i| = i$
- $\forall\, i \leq j, x_i + x_j \neq 2l.$

Its weight is defined to be
$$p := \sum_{x_i > 0} x_i$$
and its cohomological degree is $q := \#\{(i,j) \mid i \leq j \text{ and } x_i + x_j > 2l\}$.

Notation 3.6. *Given an integer x, we denote $x^+ := Max(0, x)$. Therefore, we have $p = \sum_i x_i^+$. The set of all $(u, v) \in \mathbb{N}^2$ such that $u \leq v$ and $x_u + x_v > 2l$ will be denoted by Q. The cardinality $\# Q$ will be denoted by q. Moreover we adopt the following convention: if $\mathcal{C}$ is a condition on (u, v), then $Q(\mathcal{C})$ will denote the subset of Q consisting of pairs satisfying $\mathcal{C}$. For example, given an integer v_0, the set $Q(v = v_0)$ consists of all pairs (u, v) in Q such that $v = v_0$.*

Since this excludes the case of Y being a projective space or a quadric, which occurs when $n \leq 2$, the first part of Theorem 3 amounts to the following proposition is this case :

Proposition 3.7 (First part of Theorem 3 for Lagrangian Grassmannians)**.** *Let (x_i) be an l-admissible C_n-sequence of weight p and cohomological degree q, with $n \geq 3$. Then*
$$l + q \geq \frac{2p}{n}\,,$$
with equality occurring if and only if $x_i = i, l = n + 1$ and $q = 0$.

Proof. Let $t = \#\{i \mid x_i > l\}$. Snow classified the cases where $l = 1$ [Sno88, Theorem 2.2]. We then have $p = \frac{t(t+1)}{2}$ and $q = t^2$. Since $t \leq \frac{n}{2}$, we have
$$\frac{2p}{n} = \frac{2t(t+1)}{n} \leq t + 1 \leq t^2 + 1 = q + l\,.$$
Moreover, if the equality holds, then $t = 1$ and thus $n = 2$, but this value of n has been excluded. Therefore, the proposition is true in this case.

We now assume that $l \geq 2$. Given i, j, if $x_i > l$ and $x_j > l$, then evidently $x_i + x_j > 2l$. Therefore,
$$q \geq \frac{t(t+1)}{2} \geq 2t - 1\,.$$
On the other hand, we have $p \leq \frac{l(l-1)}{2} + tn$. If $2p \geq (l+q)n$, then
$$l(l-1) + 2tn \geq (2t + l - 1)n\,.$$

Since $l > 1$, this implies $n \leq l$, and so $q = 0$.

If $l = n$, then $x_n = -n$, so we have $2p \leq n(n-1)$, therefore, $\frac{2p}{n} \leq n - 1 < l$, and the proposition is true.

If $l > n$, since $2p \leq n(n+1)$, we get that $\frac{2p}{n} \leq n + 1 \leq l$, and if the equality holds then $l = n + 1$ and $p = \frac{n(n+1)}{2}$. □

We now prove the second part of Theorem 3 :

Proposition 3.8 (Second part of Theorem 3 for Lagrangian Grassmannians)**.** *Let (x_i) be an l-admissible C_n-sequence of weight p and cohomological degree q, with $n \geq 3$ and $q > 0$. Then*

$$l + q \leq p \,.$$

Moreover, the equality $p = q + l$ holds if and only if

$$x = (-1, -2, \dots, -l, l+1, -(l+2), -(l+3), \dots, -n)$$

with $p = l + 1$ and $q = 1$.

Proof. The proof is similar to that of Proposition 3.4. Let j be the minimal integer such that there exists $i \leq j$ with $x_i + x_j > 2l$. We have $x_j = j \geq l + 1$. We want to bound $q = \# Q$. We observe that if $(u, j) \in Q$ with $j - l \leq u < j$, then $x_u > 0$. Otherwise, $x_u = -u$ and $0 < x_u + x_j = -u + j \leq l$, contradicting the assumption that $x_u + x_j > 2l$. Hence $1 \leq x_u^+$, and therefore

$$\# Q(v = j, j - l \leq u < j) \;\leq \sum_{j-l \leq u < j} x_u^+ \,.$$

Actually, a similar inequality holds with $j - l$ replaced by $j - 2l$, but in the sequel we will use the inequality $j - l \geq 1$. In fact, we have

$$\# Q(v = j, u < j - l) \;\leq\; j - l - 1 \,.$$

Finally, $\# Q(v > j) \leq \sum_{v>j} x_v^+$. Therefore, by minimality of j, we have the inequality:

$$\begin{array}{rcl} q = & = & \# Q(v = j = u) + \# Q(v = j, j - l \leq u < j) + \# Q(v = j, u < j - l) + \# Q(v > j) \\ & \leq & 1 + \sum_{j-l \leq u < j} x_u^+ + (j - l - 1) + \sum_{v>j} x_v^+ \\ & \leq & \sum_{u<j} x_u^+ + x_j + \sum_{v>j} x_v^+ - l = p - l \,. \end{array}$$

This proves the inequality.

We now deal with the case of equality. Assume that $q = p - l$. Then, asking for equalities in the previous estimates, we find that for $j - l \leq u < j$, if $x_u > 0$, then $x_u = 1$, and for $u < j - l$, $x_u + j > 2l$ by the first inequality and $x_u < 0$ by the second. In particular, $j - l - 1 \leq 0$ and hence $j = l + 1$ for otherwise $x_{j-l-1} + j = -(j - l - 1) + j = l + 1 > 2l$. For v_0 such that $j = l + 1 < v_0$, from the equality $Q(v = v_0) = x_{v_0}^+$, we infer that if $x_{v_0} > 0$ then for all $u \leq v_0$, $x_u + x_{v_0} > 2l$. In particular, $x_{v_0-1} > 0$ and $x_{v_0-2} > 0$. By decreasing induction, we find that $x_l = l$, contradicting the l-admissibility. Hence, for $j < v < 2l$, we get $x_v < 0$. Finally, x is of the form $(-1, -2, -3, \cdots, -l, l+1, -(l+2), -(l+3), \dots, -n)$ or $(1, -2, -3, \cdots, -l, l+1, -(l+2), -(l+3), \dots, -n)$. In the second case, one has $p = l + 2$ thus $q = 2$ thus $x_1 + x_{l+1} > 2l$ thus $l = 1$. But then x is not 1-admissible since $x_1 = 1$. □

3.5. **The case of spinor Grassmannians (type D_n).** In this case Y is the so-called spinor Grassmannian, that parametrises a family of n-dimensional isotropic subspaces of $\mathbb{C}^{2n}$ equipped with a non-degenerate quadratic form. It is the homogeneous space $Y = \mathrm{SO}(2n)/\mathrm{U}(n)$. By [Sno88], the non-vanishing of $H^q(Y, \Omega_Y^p(l))$ amounts to the existence of an l-admissible D_n-sequence of weight p and cohomological degree q in the following sense :

Definition 3.9. Fix $n, l \in \mathbb{N}$ with $l > 0$. A n-uple of integers $(x_i)_{0 \leq i \leq n-1}$ will be called an *l-admissible D_n-sequence* if

- $|x_i| = i$ for all $0 \leq i \leq n-1$,
- $x_i + x_j \neq l$ for all $i < j$.

Its weight is defined to be

$$p := \sum_{x_i > 0} x_i$$

and its cohomological degree $q := \#\{(i,j) \mid i < j \text{ and } x_i + x_j > l\}$.

Remark 3.10. Observe that the only 1-admissible D_n-sequence is the sequence $(0, -1, \ldots, -n)$ with $p = q = 0$. In fact, the 1-admissibility condition leads to the implication $(x_v > 0 \Longrightarrow x_{v-1} > 0)$, and thus to $x_1 = 1$ and $x_0 + x_1 = 1$.

We continue to use Notation 3.6 except that now $Q = \{(i,j) \mid i < j \text{ and } x_i + x_j > l\}$. Since Y is not a projective space or a quadric, we have $n \geq 5$. The first part of theorem 3 amounts to the following proposition in this case :

Proposition 3.11 (First part of Theorem 3 for spinor Grassmannians)**.** *Let (x_i) be an l-admissible D_n-sequence of weight p and cohomological degree q, with $n \geq 5$. Then*

$$l + q \geq \frac{4p}{n},$$

with equality occurring if and only if $x_i = i$ and $l = 2(n-1)$.

Proof. First of all, if $x_{n-1} = -(n-1)$, let x' be the sequence of length $n-1$ with $x'_i = x_i$ for $i \leq n-2$. Then x' is evidently l-admissible. It has weight p and cohomological degree q. By induction on n, we get that $l + q \geq \frac{4p}{n-1} > \frac{4p}{n}$. Thus, in the rest of the proof, we assume that $x_{n-1} = n-1$.

Let us first assume that $l > 2(n-1)$. In this case, we have $q = 0$. Since in any case $p \leq \frac{n(n-1)}{2}$, we get that $\frac{4p}{n} \leq 2(n-1) < l + q$.

Let us now assume that $n \leq l \leq 2(n-1)$. Then, we denote by u the unique integer that satisfies the following condition

$$\#\{i \mid x_i > 0, l-n+1 \leq i \leq n-1\} = u + 1.$$

Since the sequence (x_i) is l-admissible, if for $l - n + 1 \leq i \leq n-1$ we have $x_i > 0$, then $l - n + 1 \leq l - i \leq n-1$ and $x_{l-i} < 0$. This implies that

$$n - 1 - (l - n + 1) \geq 2u,$$

that is, $l \leq 2(n-u-1)$. Since $n \leq l \leq 2n - 2u - 2$, we have $n \geq 2u + 2$. The sum of positive x_i's with $l - n + 1 \leq i \leq n-1$ can be at most $(u+1)(n-1) - \frac{u(u+1)}{2}$. Therefore, we have

$$4p \leq 4(u+1)(n-1) - 2u(u+1) + 2(l-n+1)(l-n).$$

On the other hand $q \geq u$, so introducing

$$\Delta \,:=\, (u+l)n - 2(l-n+1)(l-n) - 4(u+1)(n-1) + 2u(u+1)\,,$$

the proposition amounts to the positivity of Δ whenever $n \leq l \leq 2(n-u-1)$.

After fixing u and n, the above defined Δ is a concave function on l, so we only need to consider the values of Δ when $l = n$ and when $l = 2(n-u-1)$. When $l = n$, we get that

$$\Delta \,=\, n^2 - (4+3u)n + 2u^2 + 6u + 4\,.$$

Fixing u, the two roots of this polynomial are $n = u+2$ and $n = 2u+2$. Since we know that $n \geq 2u+2$, we have $\Delta \geq 0$ for $l = n \leq 2(n-u-1)$. For $l = 2(n-1-u)$, we have

$$\Delta \,=\, 3un - 6u(u+1)$$

Since once again $n \geq 2u+2$, we get that $\Delta \geq 0$, and hence the inequality in the proposition follows for any l such that $n \leq l \leq 2(n-u-1)$.

Moreover, we show that the equality $l + q = \frac{4p}{n}$ can only occur if $x_i = i$ and $l = 2(n-1)$. Indeed, let us assume that $\Delta = 0$. By the concavity argument, we have either $l = n$ or $l = 2(n-u-1)$. If $l = n$, we also get by the above argument that $n = 2u+2$. Since the inequality

$$4p \,\leq\, 4(u+1)(n-1) - 2u(u+1)$$

becomes an equality, we conclude that x is of the form $(-0, -1, \cdots, -u, u+1, u+2, \cdots, 2u+1)$. This implies that $q = \frac{u(u+1)}{2}$, and since $q = u$, we have $u = 1$ and $n = 4$, and the last equality contradicts the hypothesis of the proposition. If $l = 2(n-u-1)$, since $\Delta = 3un - 6u(u+1) = 0$, we have $n = 2(u+1)$ or $u = 0$. The case of $n = 2(u+1)$, $n = l$, was already dealt with earlier. Thus we have $u = 0$ and $l = 2(n-1)$. The equality

$$4p = 4(u+1)(n-1) - 2u(u+1) + 2(l-n+1)(l-n)$$

amounts to $p = \frac{(n-1)n}{2}$, so that $x_i = i$ for all i, and we are in the case of the proposition.

Let us now assume that $l < n$. We consider the sequence (x'_i) with $x'_i = x_i$ for $i < n-1$ and $x'_{n-1} = -(n-1)$. We observe that (x'_i) is l-admissible with weight $p' = p - (n-1)$ and cohomological degree q' satisfying $q' \leq q - (n-l)$. In fact, $x_i + x_{n-1} > l$ for $i < n-1-l$, and $x_{n-1-l} = n-1-l$ by l-admissibility, so that $x_{n-1-l} + x_{n-1} > l$.

By our very first argument, we have $\frac{4p'}{n-1} < d + q'$, so that $\frac{4p}{n} < d + q' + 4$. Therefore, if $q' \leq q-4$, then we are done. This is indeed the case if $n - l \geq 4$. Thus, we assume that $q' \geq q-3$, and so $n \leq l+3$. We now consider these cases.

If $n = l+3$, we have $x_{n-1} = l+2$, and so $x_2 = 2$. Since $q' \geq q-3$, we get that $x_i < 0$ for $3 \leq i \leq n-2$. Thus we have $p \leq l+5$. The inequality in the proposition is implied by the inequality $l+3 > \frac{4(l+5)}{l+3}$, which in turn is true for $l \geq 3$. Observe that the value $l = 2$ is excluded because we would then have $x_4 = 4$. Therefore, either $x_2 + x_0 = 2$ (if $x_2 = 2$) or $x_2 + x_4 = 2$ (if $x_2 = -2$).

If $n = l+2$, then there is at most one integer i such that $2 \leq i \leq n-2$ and $x_i > 0$. By admissibility, $x_1 = 1$, and therefore $x_{l-1} = -l+1$. Moreover, we have $x_l = -l$. This implies that $p \leq 2l$. The inequality of the proposition is implied by the inequality $l+2 > \frac{8l}{l+2}$, which in turn is true for $l \geq 3$.

The value $n = l+1$ would contradict l-admissibility, since we would then have $x_l = l$. $\square$

We now prove the second part :

Proposition 3.12 (Second part of Theorem 3 for spinor Grassmannians)**.** *Let (x_i) be an l-admissible D_n-sequence of weight p and cohomological degree q, with $n \geq 5$ and $q > 0$. Then*

$$l + q \leq p .$$

Moreover, the equality $p = q + l$ holds if and only if there are exactly two indices i, j such that $x_i > 0, x_j > 0$ and they satisfy the condition that either $x_i + x_j = l + 1$ (in this case $q = 1$) or x is equal to $(0, 1, -2, -3, \dots, -l, l+1, -(l+2), -(l+3), \dots, -n)$ (then $q = 2$).

Proof. Let j be the smallest integer such that there exists $i < j$ with $x_i + x_j > l$. Observe that $x_j > 0$, and by D_n-admissibility, $x_0 + x_j \neq l$ so that $j \neq l$. We first deal with the case $j > l$. The argument in this case is similar to the case of type C_n:

$$\begin{array}{rcl} q = \# Q & = & \# Q(v = j, j - l \leq u < j) + \# Q(v = j, u < j - l) + \# Q(v > j) \\ & \leq & \sum_{j-l \leq u < j} x_u^+ + (j - l) + \sum_{v > j} x_v^+ \\ & \leq & \sum_{u < j} x_u^+ + x_j^+ - l + \sum_{v > j} x_v^+ = p - l \ . \end{array}$$

If under the assumption $j > l$ the equality $p = q + l$ holds, then by the second inequality we have $x_u \leq 0$ for $u < j - l$, and by the first inequality we have $x_u^+ \leq 1$ for $j - l \leq u < j$. If $x_{j-l} = -(j-l)$, then $x_{j-l} + x_j = l$, contradicting l-admissibility. Therefore, $x_{j-l} = j - l \leq 1$, so $l - l = 1$ and $x_1 = 1$. When $v > j$, the first inequality leads to the implication $(x_v > 0 \Longrightarrow \forall\, u < v, x_u + x_v > l)$. Assuming the existence of a $v > j$ such that $x_v > 0$, we get that $x_{v-1} > 0$ and $x_{v-2} > 0$ because $l > 1$ (see Remark 3.10). By descending induction, this would lead to $x_{j-1} > 0$. Then $j - 1 = 1$, hence $j = 2$, $l = 1$, which is a contradiction. Thus, if $v > j$, then $x_v < 0$. Hence $x = (0, 1, -2, -3, \cdots, -l, l+1, -(l+2), -(l+3), \dots, -n)$ (with $p = l + 2$ and $q = 2$).

Let us now assume that $j < l$, and let i be the largest integer such that $i < j$ and $x_i + x_j > l$. Note that $x_i > 0$ and $x_j > 0$, and by maximality of i we have $x_k < 0$ for $i < k < j$. We have

$$\begin{array}{rcl} q = \# Q & = & 1 + \# Q(v = j, u < i) + \# Q(v > j) \\ & \leq & 1 + \sum_{u < i} x_u^+ + \sum_{v > j} x_v^+ \\ & \leq & (x_i + x_j - l) + \sum_{u < i} x_u^+ + \sum_{v > j} x_v^+ = p - l \ . \end{array}$$

The inequality is proved. In the case of $q = p - l$, for $1 \leq u < i$ we have $x_u^+ \leq 1$ by the first inequality. By the second inequality we have $x_i + x_j = l + 1$. Using the descending induction argument, if there is a $v > j$ such that $x_v > 0$, then $x_l > 0$, and $x_0 + x_l = l$ contradicting the admissibility. The only positive entries are among x_1, x_i, x_j. In this case, since $i + j = l + 1$, we have $Q = \{(i, j)\}$, so $q = 1$, $p = l + 1$ and $x_1 < 0$. □

3.6. **The exceptional cases (type E_6 or E_7).** Now Y is homogeneous under a group of type E_6 (case $EIII$) or E_7 (case $EVII$). In the first case, we have $\dim(Y) = 16$ and $c_1(Y) = 12$. In the second case, we have $\dim(Y) = 27$ and $c_1(Y) = 18$. These values are well-known to the specialists; several arguments for the computation of c_1 can be found at the end of Section 2.1 in [CMP08].

From Tables 4.4 and 4.5 in [Sno88] we conclude that the inequalities we are looking for hold :

Proposition 3.13 (Theorem 3 for the exceptional cases)**.** *Let Y be a Hermitian symmetric space of type $EIII$ or $EVII$. Let l, p, q be integers with $l > 0, p > 0$, and such that*

$$H^q(Y, \Omega_Y(l)) \neq 0 .$$

Then, $p\, \frac{c_1(Y)}{\dim(Y)} \leq l + q$. Equality implies that $p = \dim(Y), l = c_1(Y)$ and $q = 0$.

Assume moreover that $q > 0$. Then $l + q \leq p$.

4. Restriction to a hypersurface with an increase of the Picard group

In this section, we assume that we are given a compact irreducible Hermitian symmetric space Y as in Section 2. We assume $X \subset Y$ is a smooth divisor but we do not make any assumption on $\mathrm{Pic}(X)$.

4.1. **Another argument for general complete intersection.** In this section, we want to get rid of the assumption on the Picard group. This can be done at the cost of considering only general complete intersections.

Let $V = \Gamma(Y, \mathcal{O}_Y(1))^*$ be the minimal homogeneous embedding of Y, so that $Y \subset \mathbb{P}V$. Let

$$S = \mathbb{P}S^{h_1}V^* \times \ldots \times \mathbb{P}S^{h_c}V^* ,$$

and let $Z \subset Y \times S$ be the universal family of complete intersections defined by

$$(x, ([H_1], \ldots, [H_c])) \in Z \iff \forall i, H_i(x) = 0 .$$

Thus we have morphisms $p : Z \to Y$ and $q : Z \to S$ such that for generic $s \in S$, the inverse image $Y_s := q^{-1}(s)$ is a complete intersection in Y.

To prove by contradiction, assume that semistability does not hold. We will use the relative Harder-Narasimhan filtration relative to $q : Z \to S$ [HL10, Theorem 2.3.2] (the idea of using this relative version appears e.g. in the proof of [HL10, Theorem 7.1.1]). There is a birational projective morphism $f : T \to S$ which induces a commutative diagram

$$\begin{tikzcd}
g^*p^*\Omega_Y \arrow[d] & p^*\Omega_Y \arrow[d] & \\
Z_T \arrow[r, "g"] \arrow[d] & Z \arrow[r, "p"] \arrow[d, "q"'] & Y \\
T \arrow[r, "f"'] & S &
\end{tikzcd}$$

and there is a filtration of $g^*p^*\Omega_Y$, which induce for a generic point $s \in S$ the Harder-Narasimhan filtration of $\Omega_{Y|Y_s}$. We denote by $\mathcal{F}$ the first term of this filtration and by k its rank. The rank one reflexive subsheaf $\det \mathcal{F} := (\bigwedge^k \mathcal{F})^{\star\star}$ of $p^* \bigwedge^k \Omega_Y$ is invertible. Since S is smooth, f is an isomorphism in codimension 1, so also g, and since Z is also smooth, $\det \mathcal{F}$ defines a line bundle on Z and it is a subsheaf of $\bigwedge^k p^*\Omega_Y$. We will denote $\det \mathcal{F}$ by $\mathcal{L}$. Now, p is a locally trivial morphism with fibres isomorphic to products of projective spaces, so $\mathrm{Pic}(Z) \simeq \mathrm{Pic}(Y) \times \mathrm{Pic}(S)$, and $\mathcal{L}$ can be expressed as $p^*\mathcal{L}_Y \otimes q^*\mathcal{L}_S$, for some line bundles $\mathcal{L}_Y \in \mathrm{Pic}(Y)$ and $\mathcal{L}_S \in \mathrm{Pic}(S)$.

Let d be the integer such that $\mathcal{L}_Y \simeq \mathcal{O}_Y(-d)$, and let $X = Y_s$. Given $s \in S$, we have $\mathcal{L}_{|X \times \{s\}} \simeq \mathcal{L}_{Y|X}$, and hence for generic $s \in S$, this yields an injection of sheaves $\mathcal{O}_Y(-d)_{|X} \subset p^*\Omega_Y$.

Let $h = h_1 \ldots h_c$, we have :

$$\begin{aligned}
\mu(\Omega_{Y|X}) &= \frac{\mathcal{O}_X(1)^{\dim(X)-1} \cdot K_Y}{\operatorname{rank} \Omega_{Y|X}} = -\frac{c_1(Y)}{\dim(Y)} \cdot h \cdot \deg Y \\
\mu(\mathcal{F}_{|X}) &= \frac{\mathcal{O}_X(1)^{\dim(X)-1} \cdot \det \mathcal{F}}{\operatorname{rank} \mathcal{F}} = -\frac{d}{p} \cdot h \cdot \deg Y .
\end{aligned}$$

Since $\mathcal{O}_Y(-d)_{|X} \subset \Omega_{Y|X}$, it follows that

$$H^0(X, Hom(\mathcal{O}_Y(-d)_{|X}, \Omega^k_{Y|X})) = H^0(X, \Omega^k_{Y|X}(d))$$

does not vanish. Using this and previous results we deduce the inequality

$$\mu(\mathcal{F}_{|X}) \, < \, \mu(\Omega_{Y|X}) \, , \ \ i.e., \ \ d \, > \, k \, \frac{c_1(Y)}{\dim(Y)} \, .$$

This contradicts the construction of $\mathcal{L}$ as the determinant of the first term of the relative Harder-Narasimhan filtration.

We get the following adaptation of theorems 1 and 2.

Theorem 4. *Let Y be a compact irreducible Hermitian symmetric space. Let X be a general positive-dimensional complete intersection in Y. If Y is neither a projective space nor a quadric, then the restriction of Ω_Y to X is semistable.*

If Y is a smooth quadric or a projective space, assume that none of the hypersurfaces H_i is linear. Then the restriction of Ω_Y to X is semistable.

4.2. **Results for all smooth divisors.** Recall the theorem of Langer in our setting.

Theorem ([Lan04, Theorem 5.2])**.** *Consider a compact irreducible Hermitian symmetric space Y, not isomorphic to $\mathbb{Q}^2$. Consider the $\mathcal{O}_Y(1)$-stable vector bundle Ω_Y of rank r on Y. Let X be a smooth divisor in the complete linear system $|\mathcal{O}_Y(h)|$. If*

$$h > \frac{r-1}{r}\left[2rc_2(\Omega_Y) - (r-1)c_1^2(\Omega_Y)\right] \cdot \mathcal{O}_Y(1)^{\dim Y - 2} + \frac{1}{r(r-1)\deg \mathcal{O}_Y(1)} \, ,$$

then $\Omega_{Y|X}$ is $\mathcal{O}_Y(1)$-stable.

Remark 4.1. In the cases of $\mathbb{P}^3$ and $\mathbb{Q}^3$, as $r > 2$, Langer noticed in [Lan04, Remark 5.3.2] that the inequality $h > \frac{r-1}{r}\left[2rc_2(\Omega_Y) - (r-1)c_1^2(\Omega_Y)\right] \cdot \mathcal{O}_Y(1)^{\dim Y - 2}$ is enough to ensure stability by restriction. Hence, the bounds of Langer are

- for $\mathbb{P}^2$, $h > 2$,
- for $\mathbb{P}^3$, $h > 8/3$,
- for $\mathbb{Q}^3$, $h > 8$.

With some obvious exceptions, we get the stability of $\Omega_{Y|X}$:

Theorem 5. *Let Y be a compact irreducible Hermitian symmetric space of dimension 2 or 3. Let $X \subset Y$ be a smooth divisor.*

- *Take $Y = \mathbb{P}^2$. Then $\Omega_{Y|X}$ is semi-stable if $\deg X \geq 2$.*
  If $\deg X \geq 3$, then $\Omega_{Y|X}$ is stable.
- *If $Y = \mathbb{P}^3$, assume that $\deg X \geq 2$. Then $\Omega_{Y|X}$ is stable.*
- *If $Y = \mathbb{Q}^2$, then $\Omega_{Y|X}$ is semi-stable but not stable.*
- *Take $Y = \mathbb{Q}^3$. If $\deg X = 1$, then $\Omega_{Y|X}$ is semi-stable.*
  If $\deg X = 2$, then $\Omega_{Y|X}$ is stable.
  If $\deg X \geq 9$, then $\Omega_{Y|X}$ is stable.

These cases will be considered in the next subsections.

4.3. **The case of $\mathbb{P}^2$.** Recall the Euler sequence on $\mathbb{P}^2$

$$0 \longrightarrow \Omega_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}{}^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0\,.$$

For a smooth conic C, because each section of $H^0(C, \mathcal{O}(1)_{|C})$ is a restriction of a sections on $\mathbb{P}^2$, the rank 2 vector bundle $\Omega_{\mathbb{P}^2}(1)_{|C}$ of degree -2 has no section. Therefore, $\Omega_{\mathbb{P}^2}(1)_{|C}$ is isomorphic to the direct sum of two line bundles of degree -1 on the rational curve C. Consequently, it is semi-stable and not stable.

Let C be a curve of degree $d \geq 3$ in $\mathbb{P}^2$. We could apply Langer theorem to get the stability of $\Omega_{\mathbb{P}^2}(1)_{|C}$, but we give an elementary argument.

As subsheaves of $\Omega_{\mathbb{P}^2}(1)_{|C}$ are subsheaves of the semi-stable sheaf $\mathcal{O}_{\mathbb{P}^2}{}^{\oplus 3}$ of zero slope, they are of non-positive slope. Consider a torsion-free (hence locally free) sheaf L on the curve C such that $0 \geq \mu(L) \geq \mu(\Omega_{\mathbb{P}^2}(1)_{|C})$ that is

$$0 \leq \deg L^\star \leq d/2.$$

We intend to show that there is no coherent sheaf homomorphism from L to $\Omega_{\mathbb{P}^2}(1)_{|C}$.

In view of the long exact sequence of cohomologies associated with the Euler sequence

$$0 \longrightarrow H^0(C, L^\star \otimes \Omega_{\mathbb{P}^2}(1)) \longrightarrow H^0(C, L^\star)^{\oplus 3} \xrightarrow{\epsilon} H^0(C, L^\star \otimes \mathcal{O}_{\mathbb{P}^2}(1))\,, \tag{4.1}$$

this amounts to proving the injectivity of the last map ϵ.

Lemma 4.2. $h^0(C, L^\star) \leq 1$.

Proof. If $d = 3$, then $\deg L^\star = 0$ or 1, and so $h^0(L^\star) = 0$ or 1 on the elliptic curve C. If $d \geq 4$, then the plane curve C of degree d does not have any g^1_m for $m \leq d-2$ (see [ACGH85, page 56]). In fact, in such a pencil there would be a reduced divisor D of degree m. The canonical map Φ_K of C is an embedding and the geometric form of Riemann-Roch theorem yields $1 \leq r(D) = m - 1 - \dim \Phi_K(D)$, in other words the dimension of the projective subspace generated by $\Phi_K(D)$ satisfies the inequality $\dim \Phi_K(D) \leq m-2$. But the codimension

$$h^0(\mathbb{P}^2, \mathcal{O}(d-3)) - h^0(\mathcal{I}_D(d-3))$$

of the space of divisors linearly equivalent to $\mathcal{O}(d-3)$ passing through D is the codimension of hyperplanes in $|K|$ passing through $\Phi_K(D)$ which is

$$h^0(|K|, \mathcal{O}(1)) - h^0(\mathcal{I}_{\Phi_K(D)}(1)) = \dim \Phi_K(D) + 1 \leq m-1\,.$$

This contradicts the fact that the $m \leq d-2$ distinct points of D impose independent conditions on curves of degree $d-3$. $\square$

The map ϵ in (4.1) reads

$$(as, bs, cs) \longmapsto asX + bsY + csZ \;=\; sl(X, Y, Z)\,,$$

where (X, Y, Z) are homogeneous coordinates on $\mathbb{P}^2$, while (s) is a basis of $H^0(C, L^\star)$ and l is a linear form in (X, Y, Z) not zero on C for (a, b, c) non zero. If $\epsilon(a, b, c) = 0$, then $l = 0$ and hence $(a, b, c) = 0$. This proves the injectivity of ϵ and the stability of $\Omega_{\mathbb{P}^2}(1)_{|C}$ for $d \geq 3$. This argument was suggested to us by Frédéric Han.

4.4. **The case of $\mathbb{P}^3$.** We assume that $X \subset \mathbb{P}^3$ is a smooth divisor of degree $d > 1$. In this case, as discussed in Remark 4.1, the bound in Langer's theorem is $\frac{8}{3}$. Therefore, we conclude that the restriction $T\mathbb{P}^3_{|X}$ is stable if $d \geq 3$.

Assume that $d = 2$. In this case $X = \mathbb{P}^1 \times \mathbb{P}^1$.

Let

$$P \subset \mathrm{SL}(2)$$

be the parabolic subgroup defined by the lower triangular matrices. The quotient

$$M := \mathrm{SL}(2)/P$$

is isomorphic to $\mathbb{P}^1$. For $i = 1, 2$, let

$$p_i : M \times M \longrightarrow M$$

be the projection to the i-th factor. Consider the line bundle

$$\mathcal{L} := (p_1^*\mathcal{O}_{\mathbb{P}^1}(1)) \otimes (p_2^*\mathcal{O}_{\mathbb{P}^1}(1)) \longrightarrow M \times M .$$

It is very ample. Let

$$\varphi : M \times M \longrightarrow \mathbb{P} := \mathbb{P}(H^0(M \times M, \mathcal{L})) \tag{4.2}$$

be the corresponding embedding. Note that $\dim \mathbb{P} = 3$.

Up to an automorphism of $\mathbb{P}^3$, the embedding ι of X in $\mathbb{P}^3$ coincides with φ in (4.2). Therefore, to prove that $\iota^*T\mathbb{P}^3_{\mathbb{C}}$ is stable it suffices to show that $\varphi^*T\mathbb{P}$ is stable with respect to the polarisation $\mathcal{L}$ on $M \times M$.

We have degree$(\varphi^*T\mathbb{P}) = 8$. In particular, degree$(\varphi^*T\mathbb{P})$ is coprime to rank$(\varphi^*T\mathbb{P}) = 3$. Therefore, $\varphi^*T\mathbb{P}$ is stable if it is semistable.

Assume that $\varphi^*T\mathbb{P}$ is not semistable. Let

$$F \subsetneq \varphi^*T\mathbb{P} \tag{4.3}$$

be the first nonzero term of the Harder–Narasimhan filtration of $\varphi^*T\mathbb{P}$.

Consider the left-translation action of SL(2) $\times$ SL(2) on

$$\mathrm{SL}(2) \times \mathrm{SL}(2)/(P \times P) = (\mathrm{SL}(2)/P) \times (\mathrm{SL}(2)/P) = M \times M .$$

The left-translation action of SL(2) on $\mathrm{SL}(2)/P = M$ has a natural lift to an action of SL(2) on $\mathcal{O}_{\mathbb{P}^1}(1)$. Using it, we get a lift of the action of SL(2) $\times$ SL(2) on $M \times M$ to the line bundle $\mathcal{L}$. This action on $\mathcal{L}$ produces an action of SL(2) $\times$ SL(2) on $\mathbb{P} := \mathbb{P}(H^0(M \times M, \mathcal{L}))$. The map φ in (4.2) is SL(2) $\times$ SL(2)–equivariant.

The action of SL(2) $\times$ SL(2) on $T\mathbb{P}$ produces an action of SL(2) $\times$ SL(2) on $\varphi^*T\mathbb{P}$. This action is a lift of the action of SL(2) $\times$ SL(2) on $M \times M$. From the uniqueness of the Harder–Narasimhan filtration it follows that the action of SL(2) $\times$ SL(2) on $\varphi^*T\mathbb{P}$ leaves the subsheaf F in (4.3) invariant. In particular, F is a subbundle of $\varphi^*T\mathbb{P}$, as the action of SL(2) $\times$ SL(2) on $M \times M$ is transitive.

Let $\mathcal{V}$ be the trivial vector bundle over M with fibre $H^0(M, \mathcal{O}_{\mathbb{P}^1}(1)) = \mathbb{C}^{\oplus 2}$. Consider the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(1) \otimes_{\mathbb{C}} H^0(M \times M, \mathcal{L}) \longrightarrow T\mathbb{P} \longrightarrow 0$$

on $\mathbb{P}$. Its pullback by φ in (4.2) is the following:

$$0 \longrightarrow \mathcal{O}_{M \times M} \longrightarrow \mathcal{L} \otimes (p_1^*\mathcal{V}) \otimes (p_2^*\mathcal{V}) \xrightarrow{\gamma} \varphi^*T\mathbb{P} \longrightarrow 0 . \tag{4.4}$$

Each of the three vector bundles in the short exact sequence in (4.4) is equipped with an action of $\mathrm{SL}(2) \times \mathrm{SL}(2)$, and all the homomorphisms in (4.4) are $\mathrm{SL}(2) \times \mathrm{SL}(2)$–equivariant. Since F is $\mathrm{SL}(2) \times \mathrm{SL}(2)$–invariant, for the projection γ in (4.4) we conclude that the inverse image

$$\gamma^{-1}(F) \subset \mathcal{L} \otimes (p_1^* \mathcal{V}) \otimes (p_2^* \mathcal{V})$$

is either $T_1 := \mathcal{L} \otimes (p_1^* \mathcal{O}_{\mathbb{P}^1}(-1)) \otimes (p_2^* \mathcal{V})$, or $T_2 := \mathcal{L} \otimes (p_1^* \mathcal{V}) \otimes (p_2^* \mathcal{O}_{\mathbb{P}^1}(-1))$ (note that $\mathcal{O}_{\mathbb{P}^1}(-1)$ is an equivariant subsheaf of $\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{V}$), or $T_1 + T_2$.

Indeed, this follows from the fact that the only nonzero subspaces of $\mathbb{C}^{\oplus 2}$ preserved by P are $0 \oplus \mathbb{C}$ and $\mathbb{C}^{\oplus 2}$.

From the above property of $\gamma^{-1}(F)$ it follows immediately that

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} = 2 .$$

Therefore, we have

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} = 2 < \frac{8}{3} = \frac{\mathrm{degree}(\varphi^* T\mathbb{P})}{\mathrm{rank}(\varphi^* T\mathbb{P})} .$$

But this contradicts the assumption that F is the first nonzero term of the Harder–Narasimhan filtration of $\varphi^* T\mathbb{P}$. In view of this contradiction, we conclude that $\varphi^* T\mathbb{P}$ is semistable.

4.5. **The case of $\mathbb{Q}^2$.** We consider a nondegenerate quadric Q in $\mathbb{P}^3$. Recall that it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ in such a way that $\mathcal{O}_{\mathbb{P}^3}(1)_{|Q} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$. Note that $TQ = \mathcal{O}(2,0) \oplus \mathcal{O}(0,2)$, being the sum of two line bundles of the same $\mathcal{O}_{\mathbb{P}^3}(1)_{|Q}$-degree, is $\mathcal{O}_{\mathbb{P}^3}(1)_{|Q}$-semi-stable. Its restriction $TQ_{|X}$ to a smooth curve $X \subset Q$ in any linear system $|\mathcal{O}_{\mathbb{P}^3}(d)_{|Q}| = |\mathcal{O}(d,d)|$ is the sum of two line bundles of degree $\mathcal{O}(d,d) \cdot \mathcal{O}(2,0) = \mathcal{O}(d,d) \cdot \mathcal{O}(0,2) = 2d$. Hence $TQ_{|X}$ is semi-stable for any $d \geq 1$.

4.6. **The case of $\mathbb{Q}^3$.** We now consider a nondegenerate quadric Q in $\mathbb{P}^4$ and a smooth hypersurface $S \subset Q$ of degree d. By the results in Section 3.3, the vector bundle TQ stable. The bound in Langer's theorem computed in Remark 4.1 is 8. Therefore, we conclude that the restriction $TQ_{|S}$ is stable if $d \geq 9$. We do not know if this bound is optimal. We will study low degree cases in the rest of this subsection.

4.6.1. *Estimate of the Picard group.* From the Koszul resolution of the ideal sheaf $\mathcal{I}_S$,

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-2-d) \to \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-d) \to \mathcal{I}_S \to 0$$

and the defining sequence for the structure sheaf $\mathcal{O}_S$, namely

$$0 \to \mathcal{I}_S \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_S \to 0 ,$$

we infer that $H^1(S, \mathcal{O}_S) = 0$. Therefore, the Picard group of S is discrete, and

$$\mathrm{Pic}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S).$$

We also infer that $H^2(S, \mathcal{O}_S)$ is the kernel of the surjective map

$$H^4(\mathcal{O}_{\mathbb{P}^4}(-2-d)) = S^{d-3} V^\star \xrightarrow{F_Q} S^{d-5} V^\star = H^4(\mathcal{O}_{\mathbb{P}^4}(-d))$$

gotten by contracting using the equation $F_Q \in S^2 V$ of Q, where V as before denotes $H^0(\mathbb{P}^4, \mathcal{O}(1))$. Hence, we have

$$H^2(S, \mathcal{O}_S) = \begin{cases} 0 & \text{if} \quad d = 1, 2 \\ \mathbb{C} & \text{if} \quad d = 3 \\ V^\star & \text{if} \quad d = 4. \end{cases}$$

Dimension count leads to $h^2(S, \mathcal{O}_S) = \frac{(d-1)(d-2)(2d-3)}{6}$.

From the normal sequence

$$0 \to TS \to T\mathbb{P}^4_{|S} \to \mathcal{O}(2) \oplus \mathcal{O}(d)_{|S} \to 0$$

we can compute the Euler class

$$c_2(TS) = (d^2 - 3d + 4)c_1(\mathcal{O}_{\mathbb{P}^4}(1))^2_{|S} \,,$$

the topological Euler characteristic $\chi_{top}(S) = \int_S c_2(TS) = 2d(d^2 - 3d + 4)$ and the second Betti number

$$b_2(S) = \chi_{top}(S) - (2 - 2b_1) = \chi_{top}(S) - 2 = 2(d^3 - 3d^2 + 4d - 1).$$

In particular

$$b_2(S) = \begin{cases} 2 & \text{if} \quad d = 1 \\ 6 & \text{if} \quad d = 2 \\ 22 & \text{if} \quad d = 3 \\ 62 & \text{if} \quad d = 4. \end{cases}$$

We deduce that

$$h^{1,1}(S) = \frac{d(4d^2 - 9d + 11)}{3} \ \text{ and in particular } h^{1,1}(S) = \begin{cases} 2 & \text{if} \quad d = 1 \\ 6 & \text{if} \quad d = 2 \\ 20 & \text{if} \quad d = 3 \\ 52 & \text{if} \quad d = 4. \end{cases}$$

4.6.2. *Linear sections.* For $d = 1$, the isomorphism $\mathrm{Pic}(S) = \mathbb{Z}^2$ is due to the product structure $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 4.3. *If S is a smooth linear section of the solid quadric Q, then $TQ_{|S}$ is semi-stable but not stable.*

Proof. Consider a putative destabilizing sheaf $\mathcal{F} \subset TQ_{|S}$. Assume that the rank of $\mathcal{F}$ is one. Replacing $\mathcal{F}$ by its reflexive hull, we get an exact sequence

$$0 \to L \to TQ_{|S} \to E \tag{4.5}$$

with L a line bundle and E a rank 2 vector bundle. Moreover, we have $\deg(\mathcal{F}) = \deg(L)$, thus to prove semi-stability it suffices to show that the existence of such an exact sequence implies that $\deg(L) \le \mu(TQ_{|S}) = 2$.

Let us write $S = \mathbb{P}^1 \times \mathbb{P}^1 \overset{\pi_1,\pi_2}{\to} \mathbb{P}^1$, and $L = \mathcal{O}(d_1, d_2) := \pi_1^*\mathcal{O}(d_1) \otimes \pi_2^*\mathcal{O}(d_2)$. For example, for $L = \pi_1^* T\mathbb{P}^1$, we have an exact sequence as in (4.5), and $L \simeq \mathcal{O}(2, 0)$ so $\deg(L) = 2$. In particular $TQ_{|S}$ can not be stable. The semi-stability inequality is proved in the following Lemma 4.4.

Lemma 4.4. *With the notation of the proof of Proposition 4.3, let $\mathcal{O}(d_1, d_2)$ be a subbundle of $TQ_{|S}$. Then, we have $d_1 + d_2 \le 2$.*

Proof. Since $L = \mathcal{O}(d_1, d_2)$ is assumed to be a subbundle of $TQ_{|S}$, there is a nowhere vanishing section of $L^* \otimes TQ_{|S}$. There is an exact sequence of sections on S :

$$H^0(L^* \otimes TS) \to H^0(L^* \otimes TQ_{|S}) \to H^0(L^* \otimes \mathcal{O}_S(1) \,.$$

We have $L^* \otimes \mathcal{O}_S(1) \simeq \mathcal{O}(1 - d_1, 1 - d_2)$ and $L^* \otimes TS \simeq \mathcal{O}(2 - d_1, -d_2) \oplus \mathcal{O}(-d_1, 2 - d_2)$.

Assume that $d_1 > 1$. Then $H^0(L^*(1))) = 0$ since $\pi_{1,*}L^*(1) = 0$. Thus

$$H^0(L^* \otimes TQ_{|S}) = H^0(L^* \otimes TS) = H^0(\mathcal{O}(2 - d_1, -d_2)) \,.$$

By the same argument, this space of sections is not reduced to $\{0\}$ if and only if $d_1 = 2$ and $d_2 \le 0$. Moreover, there will be nowhere vanishing sections if and only if $d_2 = 0$. We get $(d_1, d_2) = (2, 0)$ (so L is isomorphic to $\pi_1^* T\mathbb{P}^1$).

Similarly, we can deal with the case $d_2 > 1$. In the remaining cases we indeed have $d_1 + d_2 \le 2$ (asserted in the lemma). $\square$

We now finish the proof of Proposition 4.3. Consider a rank two subsheaf

$$F \subset \Omega_{Y|X} .$$

Its determinant is a line subbundle of $\Omega^2_{Y|X}$. As

$$TY(-1) = \mathcal{O}(-1)^\perp / \mathcal{O}(-1)$$

is self-dual, we have $\Omega^2_Y = K_Y \otimes TY = TY(-3) = \Omega_Y(-1)$. Hence we get an inclusion

$$\det F \otimes \mathcal{O}(1) \subset \Omega_{Y|X} .$$

We already checked that line subbundles do not destabilise $\Omega_{Y|X}$. Therefore, we have

$$2\mu(F) + 2 \le \mu(\Omega_{Y|X}) = -2 ,$$

which implies the desired semi-stability inequality

$$\mu(F) \le \mu(\Omega_{Y|X}).$$

$\square$

4.6.3. *Quadric sections.* For $d = 2$, the surface S is a Del Pezzo surface of degree 4 meaning

$$(-K_S) \cdot (-K_S) = 4$$

(see [Dol12, Definition 8.1.12]); it is known as a Segre quartic surface. We have $\mathrm{Pic(S)} = \mathbb{Z}^6$, and it is explained by the abstract description of S as the projective plane $\mathbb{P}^2$ blown-up at 5 points in general position (see[GH94, page 550], [Dol12, Proposition 8.1.25]). Recall the diagram

$$\begin{tikzcd} Bl_p(S) \arrow[r, equal, "\phi"] \arrow[d, "\mu"] & \Sigma \arrow[r, hook, "\iota"] \arrow[d, "b"] & \mathbb{P}^3 \\ S \arrow[r, "\pi"] & \mathbb{P}^2 & \end{tikzcd}$$

where μ is the blow up of S at a point p on S not on a line of S, $\iota \circ \phi$ is given by the linear system of lines in $\mathbb{P}^4$ passing through p (its image is a smooth cubic Σ), b is the blow up of $\mathbb{P}^2$ at six points, ι is given by the linear system of cubics in $\mathbb{P}^2$ passing through the blown-up six points, and π is gotten from ϕ by the universal property of blow ups. With $E = \sum_{i=1}^5 E_i$, and $\jmath : E \to S$ is the natural inclusion, we have

$$\mathcal{O}_{\mathbb{P}^4}(1)_{|S} = \pi^\star \mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_S(-E)$$

and

$$0 \to \pi^\star \Omega_{\mathbb{P}^2} \to \Omega_S \to \jmath_\star \Omega_E \to 0. \tag{4.6}$$

Proposition 4.5. *If S is a smooth quadric section of the solid quadric Q, then $\Omega_{Q|S}$ is stable.*

The proof runs through the rest of this subsection. To begin with, consider a line bundle

$$L = \pi^\star \mathcal{O}_{\mathbb{P}^2}(-a) \otimes \mathcal{O}_S(-\sum b_j E_j)$$

with an inclusion $L \subset \Omega_{Q_{|S}}$ which is seen as a nonzero element of $H^0(S, Hom(L, \Omega_{Q_{|S}}))$.

$$\begin{aligned} \mu(\Omega_{Q_{|S}}) &= \frac{K_Q \cdot \mathcal{O}_{\mathbb{P}^4}(1)_{|S}}{3} = \frac{(-5+2)1 \times 2 \times 2}{3} = -4 \\ \mu(L) &= L \cdot \mathcal{O}_{\mathbb{P}^4}(1)_{|S} = -3a - \sum b_j \,. \end{aligned}$$

We first intend to show the semi-stability inequality

$$3a + \sum b_j \geq 4,$$

and show that equality can occur only if $L = \mathcal{O}_S(-2E_0 + 2E_j)$. By the general argument, this is ensured if L is in the restriction of the Picard group of $\mathbb{P}^4$ i.e., a multiple of $\mathcal{O}_{\mathbb{P}^4}(1)_{|S}$.

The conormal sequence for S in Q reads

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-2)_{|S} \to \Omega_{Q_{|S}} \to \Omega_S \to 0\,. \tag{4.7}$$

If $H^0(Hom(L, \mathcal{O}_{\mathbb{P}^4}(-2)_{|S})) \neq 0$, then $\mu(L) \leq \mu(\mathcal{O}_{\mathbb{P}^4}(-2)_{|S}) = -8 < \mu(\Omega_{Q_{|S}})$, and the desired inequality is proved, in its strict version.

From now on, we will assume that

$$H^0(Hom(L, \mathcal{O}_{\mathbb{P}^4}(-2)_{|S})) = 0\,.$$

Hence a non-zero element in $H^0(S, Hom(L, \Omega_{Q_{|S}}))$ gives a non-zero element in

$$H^0(S, Hom(L, \Omega_S)) = H^0(S, \Omega_S \otimes \mathcal{O}_{\mathbb{P}^2}(a) \otimes \mathcal{O}_S(\sum b_j E_j))\,.$$

In particular,

$$H^0(S - \cup E_j, \Omega_S \otimes \pi^\star \mathcal{O}_{\mathbb{P}^2}(a)) = H^0(\mathbb{P}^2 - \cup p_j, \Omega_{\mathbb{P}^2}(a)) = H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(a)) \neq 0$$

and from the Euler sequence with $V := H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$

$$0 \to \mathcal{O}_{\mathbb{P}^2}(a-3) \to V^\star \otimes \mathcal{O}_{\mathbb{P}^2}(a-2) \to T_{\mathbb{P}^2}(a-3) = \Omega_{\mathbb{P}^2}(a) \to 0$$

this leads to

$$a \geq 2.$$

Let j be an integer between 1 and 5. To make use of the sequence (4.6), inspired by [Fah89], we consider the rational section of $\Omega_{\mathbb{P}^2}$ given in homogeneous coordinates $[X : Y : Z]$ such that the blown up point p_j is $[0 : 0 : 1]$ by

$$\omega_j := \frac{XdY - YdX}{Z^2} = \left(\frac{X}{Z}\right)^2 d\left(\frac{Y}{X}\right).$$

Its pull-back $\pi^\star \omega_j$ on S has poles only along the strict transform E_0 of the line $(Z = 0)$ with order two, and vanishes with multiplicity two along the exceptional divisor E_j above p_j

$$\pi^\star \omega \in H^0(S, \Omega_S \otimes \mathcal{O}_S(2E_0 - 2E_j)).$$

If $H^0(Hom(L, \mathcal{O}_S(-2E_0 + 2E_j))) \neq 0$, then $\mu(L) \leq \mu(\mathcal{O}_S(-2E_0 + 2E_j)) = -4 = \mu(\Omega_{Q_{|S}})$ with equality if and only if L is isomorphic to $\mathcal{O}_S(-2E_0 + 2E_j)$.

We assume from now on that $H^0(S, Hom(L, \Omega_S)) \neq 0$ and that for all j, $H^0(Hom(L, \mathcal{O}_S(-2E_0 + 2E_j))) = 0$. The rational form $\pi^\star\omega_j$ yields the sequence

$$0 \to \mathcal{O}_S(-2E_0 + 2E_1) \to \Omega_S \to K_S \otimes \mathcal{O}_S(2E_0 - 2E_j)$$

and after a twist by $L^\star$ a map

$$H^0(Hom(L, \Omega_S)) \to H^0(L^\star \otimes K_S \otimes \mathcal{O}_S(2E_0 - 2E_j))$$

that is injective and gives a curve C_j in the linear system $|L^\star \otimes K_S \otimes \mathcal{O}_S(2E_0 - 2E_j)| = |\pi^\star\mathcal{O}_{\mathbb{P}^2}(a-1) \otimes \mathcal{O}_S((a_j - 1)E_j) \otimes \mathcal{O}_S(\sum_{k\neq j}(a_k + 1)E_k)|$.

The curves C_j are of degree $d = C_j \cdot \mathcal{O}_{\mathbb{P}^4}(1)_{|S} = \mu(L^\star)$. Denote by C'_j the sum of irreducible components of C_j that are not contracted by π and by $d'_j \leq d$ its degree.

$$C'_j \in |\pi^\star\mathcal{O}_{\mathbb{P}^2}(b_j - 1) \otimes \mathcal{O}_S((a'_j - 1)E_j) \otimes \mathcal{O}_S(\sum_{k\neq j}(a'_{jk} + 1)E_k)|.$$

As $C'_j - C_j$ consists of effective exceptional curves $b_j = a$, $a'_j \leq a_j$ and $a'_{jk} \leq a_k$.

As E_j is a line that is not a component of C'_j, $-a'_j + 1 = C'_j \cdot E_j \leq d'_j$. Hence, $a_j \geq a'_j \geq -d'_j + 1 \geq -d + 1$. The output is $d = 3a + \sum_i a_i \geq 3a - 5d + 5$ that is $d \geq \frac{a}{2} + \frac{5}{6}$.

If $a = 2$, as the only bundles L that inject into Ω_S are $\pi^\star\mathcal{O}_{\mathbb{P}^2}(-2) \otimes \mathcal{O}_S(-b_jE_j)$ with $b_j \leq 2$, of degree $-6 + b_j$, we reach the desired inequality.

We now assume $a \geq 3$ and $3 \leq d \leq 4$.

Assume that for some j, $C'_j \cdot E_j \geq 3$. Then $d'_j \geq 3$. We first observe that C'_j cannot be a plane curve. In fact, if $C'_j \subset \mathbb{P}^2$, then since S contains C'_j and is the intersection of two quadrics, it has to contain $\mathbb{P}^2$ which is absurd. There are now two cases: if $d'_j = 3$, then choosing a point $x \in C'_j \setminus E_j$ and the plane $\mathbb{P}^2$ generated by E_j and x, we get $\#\{\mathbb{P}^2 \cap C'_j\} \geq 4$, hence a contradiction.

If $d'_j = 4$, if the linear span of C'_j is the whole $\mathbb{P}^4$ then we can choose a line ℓ secant to C'_j on two points and disjoint from E_j, and we consider the 3-plane $\mathbb{P}^3$ generated by ℓ and E_j. Since $\#\{\mathbb{P}^3 \cap C'_j\} \geq 5$ we get a contradiction in this case. If the linear span $< C'_j >$ of C'_j is a 3-plane then $< C'_j > \cap S$ contains $C'_j \cup E_j$, of degree 5, a contradiction.

Thus, for all j, we have $C'_j \cdot E_j \leq 2$. Therefore $-a'_j + 1 \leq 2$, $a'_j \geq -1$. Hence, $d = 3a + \sum a_i \geq 3 \times 3 + 5(-1) = 4$, with equality occurring if and only if L is isomorphic to $\pi^\star\mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_S(\sum E_i) = \mathcal{O}_{\mathbb{P}^4}(-1)_{|S}$. However, we already know that this is not possible.

We now prove stability, namely that $L = \mathcal{O}_S(-2E_0 + 2E_j)$ is not a subsheaf of Ω_Q. Let $C = 2E_0 - \sum E_i$ be the class of the strict transform of the conic in $\mathbb{P}^2$ passing through the five points p_i. Observe that $L = \mathcal{O}_{\mathbb{P}^4}(-2) \otimes \mathcal{O}_S(2C + 2E_j)$, thus a section of $L^* \otimes \Omega_Q$ is a section of $\Omega_Q(2)_{|S}$ that vanishes at order 2 along E_j and C.

The proof of Proposition 4.5 will therefore be complete once the following lemma is proved :

Lemma 4.6. *Let $s \in H^0(S, \Omega_Q(2)_{|S})$ a non-vanishing section, and let Δ_1, Δ_2 be two secant lines. Then s does not vanish at order two along Δ_1 and Δ_2.*

We will prove this lemma after some preliminary results. First, let us denote by Q_2 a quadric cutting out S in Q. By simultaneous reduction of quadratic forms, we may assume that the quadric Q is defined by the identity matrix I and Q_2 by some diagonal matrix D_2.

Since $H^1(S, \mathcal{O}_S) = 0$, the section s lifts to a section $\tilde{s} \in H^0(S, \Omega_{\mathbb{P}^4}(2)_{|S})$. We will have to consider affine cones: let $U = \mathbb{C}^5 \setminus \{0\}$ and let $p : U \to \mathbb{P}^4$. Whenever $Z \subset \mathbb{P}^4$ is a subvariety, we denote by $\widehat{Z} = p^{-1}(Z)$ its affine cone. The section $\tilde{s}$ defines a section $\widehat{s} \in H^0(\widehat{S}, \Omega_{U|\widehat{S}})$, which can be written as $\widehat{s} = \sum_{i,j} a_{i,j} Z_i dZ_j$ (Z_j denotes the j-th coordinate function on $\mathbb{C}^5$). We denote by A the matrix $(a_{i,j})$. Since $\widehat{s}$ is the pullback of the section $\tilde{s}$, we have :

Fact 4.7. *$A + {}^t A$ belongs to the span of I and D_2.*

We want to understand the scheme-theoretic vanishing locus of s. As a set, it is described by :

Fact 4.8. *Let $u \in \widehat{S}$ and $x = [u] \in S$. Then $s(x) = 0$ if and only if u is an eigenvector of A.*

Proof. The quadratic form Q yields an identification of $\mathbb{C}^5$ with its dual. Moreover, $\widehat{s}(u)$ identifies in the basis dZ_j to the column vector Au. Since the coordinates have been chosen so that the matrix of Q is I, the tangent space of $\widehat{Q}$ at u has equation u itself. Thus $s(x)$ vanishes if and only if these two linear forms define the same hyperplane, in other words if and only if Au is a multiple of u. $\square$

At first order, the vanishing of s is characterized by :

Fact 4.9. *Let $x = [u] \in S$ such that $Au = 0$ and $u \notin \mathrm{Im}(A)$. Let $X = [U] \in T_x S$, with $U \in T_u \widehat{S}$. Then, the derivative $ds_x(X)$ vanishes if and only if $AU = 0$.*

Proof. As the proof of Fact 4.8 shows, $\Omega_{\widehat{Q},x}$ identifies with $\mathbb{C}^5/\mathbb{C} \cdot u$. The statement then follows from the fact that $ds_x(X) = AX \in \mathbb{C}^5/\mathbb{C} \cdot u \simeq \Omega_{\widehat{Q},x}$. $\square$

We now prove Lemma 4.6. Let π be the plane generated by Δ_1 and Δ_2. Since s vanishes along Δ_1 and Δ_2, by Fact 4.8, $\widehat{\pi}$ must be included in an eigenspace of A. Replacing A by $A - \lambda \cdot I$ does not change the section s, thus we can assume that $\widehat{\pi} \subset \ker A$. Therefore the rank of A is at most 2.

Assume first that A has rank 2. Let $x = [u] \in (\Delta_1 \cup \Delta_2) \setminus \mathbb{P}\mathrm{Im} A$. Since s vanishes at order two along $\Delta_1 \cup \Delta_2$, by Fact 4.9, we have $T_u \widehat{S} \subset \ker A$, and so equality of these subspaces. Since we may assume that x is not the intersection point $\Delta_1 \cup \Delta_2$, we get a contradiction with the following fact :

Fact 4.10. *We have $S \cap \pi = \Delta_1 \cup \Delta_2$. For $x \in \Delta_1 \setminus \Delta_2$, $\overline{T_x S} \neq \pi$.*

Here, $\overline{T_x S} \subset \mathbb{P}^4$ denotes the embedded tangent space.

Proof. Let Q' be any quadric containing S. We have $Q' \cap \pi = \Delta_1 \cup \Delta_2$ or $Q' \cap \pi = \pi$, for degree reasons. The first point follows. Assume now that $x \in \Delta_1 \setminus \Delta_2$ and that $\overline{T_x S} = \pi$. Let ℓ be a line through x and a point y in $\Delta_2 \setminus \Delta_1$. Once again, if Q' is a quadric containing S, then $\ell \cap Q'$ has multiplicity at least 2 at x ($\ell \subset \pi = \overline{T_x S} \subset \overline{T_x Q'}$) and one at y, thus $\ell \subset Q'$. This implies that $\ell \subset S$, contradicting the first point of the Fact. $\square$

Assume now that A has rank 1. We will use the following observation :

Fact 4.11. *Let B be a square matrix which is the sum of an alternate matrix and a diagonal matrix. Assume that $\mathrm{rk} B = 1$. Then, up to a permutation of the rows and columns, B can be written as a bloc-diagonal matrix $\begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix}$, with β a rank 1 matrix of order 2.*

Proof. Write $B = (b_{i,j})$. Since a coefficient of B is non zero, a diagonal coefficient of B must be non zero, and assume that $b_{1,1} \neq 0$. If all the other diagonal coefficients are 0, then we have $b_{i,j} = 0$ for $(i,j) \neq (1,1)$ and the fact is true. In the other case, assume that $b_{2,2} \neq 0$. We have $b_{1,3} + b_{3,1} = b_{2,3} + b_{3,2} = b_{1,2} + b_{2,1} = 0$ and $b_{1,1}b_{2,3} - b_{2,1}b_{1,3} = b_{1,1}b_{3,2} - b_{3,1}b_{1,2} = 0$, with $b_{1,1}, b_{2,2}, b_{1,2}$ and $b_{2,1}$ different from 0. It follows that $b_{1,3} = b_{2,3} = b_{3,2} = b_{3,1} = 0$. Similarly, all the coefficients $b_{i,j}$ are 0 except when $i, j \leq 2$. $\square$

Now, A satisfies the hypothesis of Fact 4.11, and moreover the diagonal of A is a linear combination of I and D_2. This implies that a linear combination of I and D_2 has rank at most 2, contradicting the smoothness of S (in fact, S is smooth if and only if the quadrics in the pencil it defines all have rank at least 4). This ends the proof of Lemma 4.6.

To complete the proof of Proposition 4.5, one has to consider rank 2 subsheaves in Ω_Q. This case follows from the case of rank 1 subsheaves by the fact that Ω_Q is self-dual (see the end of the proof of Proposition 4.3).

ACKNOWLEDGEMENTS

We thank Frédéric Han and Laurent Gruson for their help in projective geometry. We thank Michel Brion for pointing out a mistake in a previous version of this paper. The first-named author is supported by a J. C. Bose Fellowship. Part of this project was done in the TATA Institute (Bombay), which we thank for its hospitality.

REFERENCES

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[Bis04] I. Biswas. On the stability of homogeneous vector bundles. *J. Math. Sci. Univ. Tokyo*, 11(2):133–140, 2004.

[BK05] Michel Brion and Shrawan Kumar. *Frobenius splitting methods in geometry and representation theory*, volume 231 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2005.

[CMP08] P. E. Chaput, L. Manivel, and N. Perrin. Quantum cohomology of minuscule homogeneous spaces. *Transform. Groups*, 13(1):47–89, 2008.

[Dol12] Igor V. Dolgachev. *Classical algebraic geometry, a modern view*. Cambridge University Press, Cambridge, 2012.

[Fah89] Rachid Fahlaoui. Stabilité du fibré tangent des surfaces de del Pezzo. *Math. Ann.*, 283(1):171–176, 1989.

[Fle84] Hubert Flenner. Restrictions of semistable bundles on projective varieties. *Comment. Math. Helv.*, 59(4):635–650, 1984.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[Har80] Robin Hartshorne. Stable reflexive sheaves. *Math. Ann.*, 254(2):121–176, 1980.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[Kim91] Sung-Ock Kim. Noether-Lefschetz locus for surfaces. *Transactions of the American Mathematical Society*, 324(1):369–384, 1991.

[KO73] Shoshichi Kobayashi and Takushiro Ochiai. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, 13:31–47, 1973.

[Lan04] A. Langer. Semistable sheaves in positive characteristic. *Ann. of Math. (2)*, 159(1):251–276, 2004.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry, I. Classical setting: line bundles and linear series*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2004.

[Lef21] Solomon Lefschetz. On certain numerical invariants of algebraic varieties with application to abelian varieties. *Trans. Amer. Math. Soc.*, 22(3):327–406, 1921.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[MR84] V. B. Mehta and A. Ramanathan. Restriction of stable sheaves and representations of the fundamental group. *Invent. Math.*, 77(1):163–172, 1984.

[Ram66] S. Ramanan. Holomorphic vector bundles on homogeneous spaces. *Topology*, 5:159–177, 1966.
[Sno86] D. M. Snow. Cohomology of twisted holomorphic forms on Grassmann manifolds and quadric hypersurfaces. *Math. Ann.*, 276(1):159–176, 1986.
[Sno88] D. M. Snow. Vanishing theorems on compact Hermitian symmetric spaces. *Math. Z.*, 198(1):1–20, 1988.
[Ume78] H. Umemura. On a theorem of Ramanan. *Nagoya Math. J.*, 69:131–138, 1978.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, BOMBAY 400005, INDIA
E-mail address: indranil@math.tifr.res.in

DOMAINE SCIENTIFIQUE VICTOR GRIGNARD, 239, BOULEVARD DES AIGUILLETTES, UNIVERSITÉ HENRI POINCARÉ NANCY 1, B.P. 70239, F-54506 VANDOEUVRE-LÈS-NANCY CEDEX, FRANCE
E-mail address: pierre-emmanuel.chaput@univ-lorraine.fr

DÉPARTEMENT DE MATHÉMATIQUES, CAMPUS DE BEAULIEU, BÂT. 22-23, UNIVERSITÉ DE RENNES 1, 263 AVENUE DU GÉNÉRAL LECLERC, CS 74205, 35042 RENNES CÉDEX, FRANCE
E-mail address: christophe.mourougane@univ-rennes1.fr